On the Hilbert polynomials and Hilbert series of homogeneous projective varieties

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Among all complex projective varieties $X \hookrightarrow \mathbb{P}(V)$, the equivariant embeddings of homogeneous varieties—those admitting a transitive action of a semi-simple complex algebraic group G—are the easiest to study. These include projective spaces, Grassmannians, non-singular quadrics, Segre varieties, and Veronese varieties. In Joe Harris' book "*Algebraic Geometry: A First Course*" [H], he computes the dimension $d = \dim(X)$ and degree $\deg(X)$ of $X \hookrightarrow \mathbb{P}(V)$ for many homogeneous varieties, in a geometric fashion.

In this expository paper we redo this calculation using some representation theory of G. We determine the Hilbert polynomial $h(t)$ and Hilbert series of the homogeneous coordinate ring of $X \hookrightarrow \mathbb{P}(V)$. Since

$$h(t) = \deg(X) \cdot \frac{t^d}{d!} + (\text{lower order terms})$$

with $d = \dim(X)$, this gives a formula for the two invariants. As a byproduct, we find that $h(t)$ is the product of linear factors over $\mathbb{Q}$.

We now state the results precisely. Fix a maximal torus T contained in a Borel subgroup B of G. The projective varieties X which admit a transitive action of G correspond to the 2^n parabolic subgroups P of G which contain B (where $n = \dim(T)$). These varieties depend only on G up to isogeny, so there is no loss of generality in assuming that G is simply-connected, and we will henceforth do so.

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The equivariant projective embeddings π_λ of $X = G/P$ into $\mathbb{P}(V)$ then correspond bijectively to the dominant weights λ for T which lie in a certain face of the closed Weyl chamber corresponding to B.

The Hilbert polynomial $h_\lambda(t)$ of the coordinate algebra of $\pi_\lambda : X \hookrightarrow P(V)$ factors as the product

$$h_\lambda(t) = \prod_\alpha (1 + c_\lambda(\alpha)t).$$

This product is taken over the set of positive roots α of G which satisfy $\langle \lambda, \alpha^\vee \rangle \neq 0$; the number d of such roots is equal to the dimension of X. In the product, $c_\lambda(\alpha)$ is the positive rational number

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where ρ is half the sum of the positive roots and $\alpha^\vee$ is the corresponding co-root (cf. [B,VI.1]). Hence

$$\deg(X) = d! \prod_\alpha \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}$$

where the product is taken over the same subset of positive roots. This simple formula for the degree was obtained by Borel and Hirzebruch [B-H, Theorem 24.10], using characteristic classes for the compact form of G.

Using the same methods we also calculate the Hilbert series of the image of the equivariant embedding corresponding to λ, yielding

$$H_\lambda(q) = h_\lambda(q\frac{\mathrm{d}}{\mathrm{d}q})\frac{1}{1-q}.$$

After sketching the proof of these results, which follow from the Borel-Weil theorem and Weyl's dimension formula, we illustrate them by calculating the degrees and Hilbert series of several equivariant embeddings. We note that the calculations for many of the Hilbert series were done with the aid of a computer. One can find the mathematica code at http://math.ucsd.edu/∼nwallach/ .

1. Equivariant embeddings

Let G be a semi-simple, simply-connected, complex algebraic group. Let $T \subset B \subset G$ be a maximal torus contained in a Borel subgroup. The choice of B determines a set of positive roots for G—those characters of T which occur in $\mathrm{Lie}(B)/\mathrm{Lie}(T)$—as well as a Weyl chamber of dominant weights in the character group of T. We say a weight λ is dominant if the integer $\langle \lambda, \alpha^\vee \rangle$ is $\geqslant 0$ for all positive roots α. Here $\alpha^\vee \in \mathrm{Lie}(T)$ is the coroot associated with α. If ρ is half the sum of the positive roots, then ρ is a dominant weight in the interior of the Weyl chamber: $\langle \rho, \alpha^\vee \rangle$ is strictly positive for all positive roots α.

Associated to every dominant weight λ for T there is an irreducible representation $V = V_\lambda$ of G over $\mathbb{C}$ with highest weight λ for B (cf. [F-H] for an introduction to this theory). Let V^* be the dual representation, and let $\langle f \rangle$ be the unique line in $V^* = \mathrm{Hom}(V, \mathbb{C})$ fixed by B; the character of T on this line is $i(\lambda)$, where i is the opposition involution of G. Let $P \supset B$ be the parabolic subgroup of G which stabilizes the line $\langle f \rangle$ in V^*, or equivalently which stabilizes the hyperplane H annihilated by f in V.

Let $\mathbb{P}(V)$ denote the projective space of *all* hyperplanes in V. This has homogeneous coordinate ring

$$\mathbb{C}[\mathbb{P}(V)] = \mathrm{Sym}^\bullet(V) = \bigoplus_{n \geqslant 0} \mathrm{Sym}^n(V)$$

Associated to λ, we have the equivariant embedding

$$\pi_\lambda : X = G/P \hookrightarrow \mathbb{P}(V)$$

defined by mapping the coset gP to the hyperplane $g(H)$. The image of π_λ is the unique closed orbit of G on $\mathbb{P}(V)$, and is a homogeneous, nonsingular projective variety [F-H,pg. 384].

2. The Hilbert polynomial and Hilbert series

We will start with some generalities on Hilbert polynomials and Hilbert series of projective embeddings. Let X be a projective variety and let $\pi : X \hookrightarrow \mathbb{P}(V)$ be an embedding of varieties over an algebraically closed field F. Let Y denote the cone in V on πX and let $F^j[\pi X]$ be the F vector space of all regular functions on Y which are homogeneous of degree j. (This vector space is denoted $S^j(X)$ in [H]. It is the quotient of the vector space $\mathrm{Sym}^j(V)$ by the subspace of homogenous polynomials of degree j which vanish on πX.)

Then the Hilbert polynomial of the embedding is the unique polynomial $h(t)$ in $\mathbb{Q}[t]$ such that

$$\dim F^j[\pi X] = h(j), \text{for all } j \gg 1.$$

The existence of such a polynomial is established in [H, Proposition 13.2], which also shows that the degree of $h(t)$ is the dimension of X.

Let $\mathcal{L}^j$ be the pullback of the Serre twist $\mathcal{O}(j)$ of the structure sheaf of $\mathbb{P}(V)$. If X is projectively normal (cf. [Ha,pg.126]), then for all $j \geqslant 0$ we have

$$\dim H^0(X, \mathcal{L}^j) = \dim F^j[\pi X].$$

Another basic invariant of the projective embedding of X is its Hilbert series. This is the formal sum

$$H(q) = \sum_{j\geq 0} \dim F^j[\pi X] q^j.$$

Serre's vanishing theorem implies that for $j \gg 1$ one has $H^i(X, \mathcal{L}^j) = 0$ for all $i > 0$. This implies that the Hilbert series is given by

$$H(q) = \sum_{j\geq 0} h(j) q^j + p(q)$$

with $p(q) \in \mathbb{Z}[q]$.

We set $\tilde{H}(q) = \sum_{j\geq 0} h(j) q^j$. If $h(t) = \sum c_k t^k$ then we see that

$$\tilde{H}(q) = \sum c_j \sum_{m\geq 0} m^j q^m.$$

The inner series has a long history (cf. [St, 1.3]). We will make a short digression on some of its properties. Obviously it is the formal expansion of

$$f_j(q) = \left(q\frac{d}{dq}\right)^j \frac{1}{1-q} = \frac{\phi_j(q)}{(1-q)^{j+1}}$$

and since $q\frac{d}{dq}$ preserves degree, $\phi_j(q)$ is a polynomial of degree j. We write

$$\phi_j(q) = \sum a_{j,i} q^i.$$

We note that $a_{j,0} = 0$. If we arrange the $a_{j,i}$ with $i = 1, ..., j$ in a triangle with jth row $a_{j,1}, ..., a_{j,j}$ we have

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 4 & & 1 & & \\
 & 1 & & 11 & & 11 & & 1 & \\
1 & & 26 & & 66 & & 26 & & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

This triangle is called Euler's triangle and it has been studied intensively. We note one property. Consider the diagonals (the second diagonal is 1,4,11,26,...) then the element in the ith diagonal and the nth row is the number of permutations with exactly i descents.

The upshot is that we have

$$H(q) = p(q) + \sum c_j \frac{\phi_j(q)}{(1-q)^{j+1}} = \frac{g(q)}{(1-q)^{d+1}}$$

with $d = \dim X$ and

$$g(q) = \sum_{j=0}^{d} c_j \phi_j(q)(1-q)^{d-j} + (1-q)^{d+1} p(q).$$

We now return to the main point of this paper. We will henceforth assume that $F = \mathbb{C}$. Fix an equivariant embedding

$$\pi_\lambda : X \hookrightarrow \mathbb{P}(V_\lambda).$$

The line bundle $\mathcal{L} = \pi^* \mathcal{O}(1)$ on X is equivariant and has sections

$$H^0(X, \mathcal{L}) = V_\lambda$$

Then $\mathcal{L}^n = \pi^* \mathcal{O}(n)$ is also equivariant, and by the theorem of the highest weight

$$H^0(X, \mathcal{L}^n) = V_{n\lambda}$$

for all $n \geqslant 0$ (cf. [B-H,p.393]).

Since the restriction homomorphism ($n \geqslant 0$)

$$\begin{array}{ccc} H^0(\mathbb{P}(V_\lambda), \mathcal{O}(n)) & \longrightarrow & H^0(X, \mathcal{L}^n) \\ \| & & \| \\ \mathrm{Sym}^n(V_\lambda) & \longrightarrow & V_{n\lambda} \end{array}$$

is G-equivariant and non-zero, and $V_{n\lambda}$ is irreducible, it must be surjective for all $n \geqslant 0$.

Hence the embedding of X is projectively normal and the homogeneous coordinate ring of the embedding is

$$\mathbb{C}[\pi_\lambda X] = \bigoplus_{n \geqslant 0} V_{n\lambda}.$$

In particular, the Hilbert polynomial $h_\lambda(t)$ of $\pi_\lambda : X \hookrightarrow \mathbb{P}(V)$ satisfies

$$h_\lambda(n) = \dim V_{n\lambda}$$

for $n \gg 0$.

But the Weyl dimension formula [S] states that

$$\dim V_{n\lambda} = \prod_{\alpha > 0} \frac{\langle n\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle},$$

where the product is taken over all positive roots α. Hence

$$\dim V_{n\lambda} = \prod_{\alpha > 0} (1 + n \cdot c_\lambda(\alpha))$$

with

$$c_\lambda(\alpha) = \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle}.$$

Therefore the polynomial

$$h_\lambda(t) = \prod_{\alpha > 0} (1 + t \cdot c_\lambda(\alpha))$$

satisfies $h_\lambda(n) = \dim V_{n\lambda}$ for all $n \geqslant 0$. This completes the determination of the Hilbert polynomial of $X \hookrightarrow \mathbb{P}(V)$ using representation theory. We note that Manivel

[M] has established some interesting results on the roots of this polynomial, for maximal parabolic subgroups P.

We now turn to the study of the Hilbert series. The fact that $h(n)$ gives the dimension of $F^n[\pi X]$ for all $n \geqslant 0$ implies that the polynomial $p(q)$ in the general Hilbert series is zero in this case. This yields the following formula for the Hilbert series

$$H(q) = \sum_{n\geq 0} \left(\prod_{\langle \lambda, \alpha^\vee \rangle > 0} \frac{\langle n\lambda + \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \right) q^n = \sum_{n \geq 0} \left(\prod_{\langle \lambda, \alpha^\vee \rangle > 0} (nc_\lambda(\alpha) + 1) \right) q^n.$$

Let $\beta_1, \ldots, \beta_d$ be an enumeration of the set of roots α such that $\langle \lambda, \alpha^\vee \rangle > 0$. Let e_j be the jth elementary symmetric function in d variables. Then we have (after a bit of manipulation and in the notation of the beginning of this section)

$$H(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) f_j(q).$$

By the above, this implies that $H(q) = \frac{g(q)}{(1-q)^{d+1}}$ with

$$g(q) = \sum_{j=0}^{d} e_j(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) \phi_j(q)(1-q)^{d-j}.$$

In particular, since $g(1) = \deg \pi_\lambda$, we have the formula

$$\deg \pi_\lambda = e_d(c_\lambda(\beta_1), c_\lambda(\beta_2), \ldots, c_\lambda(\beta_d)) \phi_d(1).$$

This agrees with the formula for the degree in the introduction since $\phi_d(1) = d!$.

There is another more geometric way of writing the above formula for $H(q)$. We note that if we consider the case of the standard Segre embedding of $(\mathbb{P}^1)^j = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ (j copies) into $\mathbb{P}(\otimes^j \mathbb{C}^2)$ then the Hilbert series is

$$\sum_{n \geq 0} (n+1)^j q^n = \frac{\phi_j(q)/q}{(1-q)^{j+1}}.$$

So the degree of this embedding is $d!$. This also says that the formula above for $H(q)$ expresses the Hilbert series of $\pi_\lambda(G/P)$ in terms of the Hilbert series of $(\mathbb{P}^1)^j$ for $j = 1, \ldots, d$. The simplest example is the case of G/B with $\lambda = \rho$. Then the formula becomes

$$H_{G/B}(q) = H_{(\mathbb{P}^1)^d}(q)$$

with d equal to the number of positive roots.

We also record the following result.

Theorem. *The Hilbert series of the embedding π_λ of G/P is*

$$\prod_{\langle\lambda,\check{a}\rangle>0} (\frac{\langle\lambda,\check{a}\rangle}{\langle\rho,\check{a}\rangle} q\frac{d}{dq} + 1)\frac{1}{1-q}.$$

3. Veronese varieties

For the first examples, we observe that the variety X remains unchanged as we scale λ by an integer $m \geqslant 1$. If $\dim(X) = d$, then

$$\deg(\pi_{m\lambda}) = m^d \cdot \deg(\pi_\lambda)$$

as every factor $c(\alpha)$ in the product for the degree is scaled by m.

We apply this to $G = SL(V)$ and $V = V_\lambda$ the standard representation. Then $X = \mathbb{P}(V) = \mathbb{P}^n$, where $\dim(V) = n+1$, and $\deg(\pi_\lambda) = 1$. Hence the Veronese embedding

$$\pi_{m\lambda} : \mathbb{P}^n \longrightarrow \mathbb{P}(\mathrm{Sym}^m V) = \mathbb{P}^{\binom{m+n}{n}-1}$$

has degree $= m^n$.

For $n = 1$, this is the rational normal curve, of degree m in $\mathbb{P}^m$. For $n = 2$ and $m = 2$ this gives the degree $(= 4)$ of the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

4. The flag variety

Another simple case is the embedding of the full flag variety $X = G/B$ using the representation V_ρ. (The dominant weight ρ is the simplest weight in the interior of the Weyl chamber; the stabilizer of its highest weight vector $\langle v_\rho \rangle$ is equal to B.)

In this case, $\dim(V_\rho) = 2^d$ by the Weyl dimension formula, where $d = \dim(X)$ is the number of positive roots. Moreover, for every positive root α we have

$$c_\rho(\alpha) = \frac{\langle \rho, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} = 1.$$

Hence $h_\rho(t) = (t+1)^d$ (the corresponding Hilbert series has been studied in section 2) and

$$\pi_\rho : X = G/B \hookrightarrow \mathbb{P}^{2^d - 1}$$

has degree $= d!$.

It is interesting to compare these results with the linear system $|2\Theta|$ on a principally polarized abelian variety A of dimension d, which maps $A \to \mathbb{P}^{2^d-1}$ with degree $2^d \cdot d!$.

5. Segre varieties

We next consider the representation of $G = SL(W) \times SL(U)$ on $V = \mathrm{Hom}(W, U) = V_\lambda$. The closed orbit X of G on $\mathbb{P}(V)$ consists of the linear maps of rank 1; this gives the Segre embedding

$$\pi_\lambda = \pi_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$$

where $n + 1 = \dim(W)$ and $m + 1 = \dim(U)$.

Let $\{e_1, \ldots, e_{n+1}\}$ be the weights for $SL(W)$ on W and $\{f_1, \ldots, f_{m+1}\}$ be the weights for $SL(U)$ on U. The highest weight of $V_\lambda = W^* \otimes U = \bigwedge^n W \otimes U$ is

$$\lambda = (e_1 + e_2 + \cdots + e_n) + f_1.$$

There are $(n+m) = d$ positive roots α with $c_\lambda(\alpha) \neq 0$:

$$\alpha = e_i - e_{n+1} \qquad i = 1, 2, \ldots, n$$

$$\alpha = f_1 - f_j \qquad j = 2, 3, \ldots, m+1.$$

Since $\rho = ne_1 + (n-1)e_2 + \cdots + e_n + mf_1 + (m-1)f_2 + \cdots + f_m$ we find

$$\begin{aligned} c_\lambda(\alpha) &= \frac{1}{(n+1-i)} && \text{in the first case} \\ &= \frac{1}{(j-1)} && \text{in the second case.} \end{aligned}$$

Hence

$$\begin{aligned} \deg(\pi_{n,m}) &= d! \prod c_\lambda(\alpha) \\ &= (m+n)! \cdot \frac{1}{n!} \cdot \frac{1}{m!} \\ &= \binom{m+n}{n}. \end{aligned}$$

For example, the degree of the Segre 3-fold

$$\pi_{1,2} : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$$

is equal to $\binom{3}{1} = 3$.

Using the same data we can compute the Hilbert series of this embedding of $\mathbb{P}^n \times \mathbb{P}^m$ for $n \leq m$ yielding

$$\frac{\sum_{1 \leq j \leq n} \binom{n}{j}\binom{m}{j} q^j}{(1-q)^{n+m+1}}.$$

6. Grassmannians

We now consider the Plucker embedding of the Grassmannian $G(k,n)$ of $(n-k)$ – planes (i.e., subspaces of codimension k) in $\mathbb{C}^n$. In this case $G = SL_n$ and $V = V_\lambda = \bigwedge^k \mathbb{C}^n$.

The highest weight λ of V is

$$\lambda = e_1 + e_2 + \cdots + e_k$$

(here and below we will be using e_i to mean the Bourbaki ϵ_i restricted to the trace 0 diagonal matrices) and there are $d = k(n-k)$ positive roots α with $c_\lambda(\alpha) = \langle \lambda, \alpha^\vee \rangle / \langle \rho, \alpha^\vee \rangle$ non-zero. We recall that

$$\rho = (n-1)e_1 + (n-2)e_2 + \cdots + e_{n-1}.$$

The relevant roots are those of the form $\alpha = e_i - e_j$ with $1 \leqslant i \leqslant k$ and $k+1 \leqslant j \leqslant n$. All of these roots have $\langle \lambda, \alpha^\vee \rangle = 1$, and we find that $c_\lambda(\alpha) = 1/(j-i)$. Hence

$$\begin{aligned} \deg(G(k,n)) = & \quad d! \prod_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant n}} \frac{1}{(j-i)} \\ = & \quad (k(n-k))! \prod_{1 \leqslant i \leqslant k} \frac{(k-i)!}{(n-i)!}. \end{aligned}$$

For example, the degree of $X = Gr(2, n+2)$ in $\mathbb{P}(\bigwedge^2 \mathbb{C}^{n+2}) = \mathbb{P}^{(n^2+3n)/2}$ is equal to

$$(2n)! \frac{1}{(n+1)!} \frac{1}{n!} = \frac{1}{n+1} \binom{2n}{n},$$

the Catalan number c_n.

The corresponding Hilbert series for X is

$$\frac{\sum_{1 \leq j \leq n} \frac{1}{n} \binom{n}{j} \binom{n}{j-1} q^{j-1}}{(1-q)^{2n+1}}.$$

The polynomial in the numerator has coefficients the Narayana numbers. If these numbers are laid out in a triangle they yield the so called Catalan triangle. For details on Narayana and Catalan polynomials see [St].

A similar case is the Lagrangian Grassmannian X of maximal isotropic subspaces (of dimension n) in a symplectic space of dimension $2n$. Here $G = Sp_{2n}$ and $V = V_\lambda = \bigwedge^n \mathbb{C}^{2n} - \bigwedge^{n-2} \mathbb{C}^{2n}$ has dimension $\frac{1}{(n+2)(n+1)}(4n+2)\binom{2n}{n}$. The highest weight is $\lambda = e_1 + e_2 + \cdots + e_n$ and there are $d = n(n+1)/2$ positive roots with $c_\lambda(\alpha)$ non-zero. These roots have the form $\alpha = e_i + e_j$ with $1 \leqslant i \leqslant j \leqslant n$. We have $c_\lambda(\alpha) = 2/(2n+2-i-j)$, so

$$deg(X) = 2^d d! \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{1}{(2n+2-i-j)}.$$

7. Two exceptional homogeneous varieties

We now consider an exceptional variety $X \hookrightarrow \mathbb{P}^{26}$ of dimension $d = 16$. Here $G = E_6$ and $V = V_\lambda$ is a minuscule representation of dimension 27. In the notation of [B] the positive roots α with $\langle \lambda, \alpha^\vee \rangle \neq 0$ have the form

$$\alpha = \frac{1}{2}\left(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu(i)} e_i\right)$$

with $\sum_{i=1}^5 \nu(i)$ even. They all satisfy $\langle \lambda, \alpha^\vee \rangle = 1$ so it suffices to compute their inner products with

$$\rho = 4(e_8 - e_7 - e_6 + e_5) + 3e_4 + 2e_3 + e_2$$

We find

$$\langle \rho, \alpha^\vee \rangle = 6 + 2(-1)^{\nu(5)} + \frac{3}{2} \cdot (-1)^{\nu(4)} + (-1)^{\nu(3)} + \frac{1}{2}(-1)^{\nu(2)}$$

$\nu(5)$	$\nu(4)$	$\nu(3)$	$\nu(2)$	$\langle\rho, \alpha^\vee\rangle$
0	0	0	0	11
0	0	0	1	10
0	0	1	0	9
0	0	1	1	8
0	1	0	0	8
0	1	0	1	7
0	1	1	0	6
0	1	1	1	5
1	0	0	0	7
1	0	0	1	6
1	0	1	0	5
1	0	1	1	4
1	1	0	0	4
1	1	0	1	3
1	1	1	0	2
1	1	1	1	1

A table of these inner products appears below, and we find

$$\begin{aligned}\deg(X) &= 16!/11!(8.7.6.5.4) \\ &= 16.15.14.13.12./8.7.6.5.4 \\ &= 78\end{aligned}$$

Is there any reason that this degree is equal to the dimension of the algebraic group E_6 which acts on X? This question also appears in [I-L].

We note that if we use the formula for the Hilbert series of this embedding and the above table we find the formula

$$\frac{1 + 10q + 28q^2 + 28q^3 + 10q^4 + q^5}{(1-q)^{17}.}$$

Similarly, for the minuscule representation V_λ of dimension 56 for the exceptional group E_7, we find that X has dimension $d = 27$ and degree $= 13110 = 2.3.5.19.23$ in $\mathbb{P}^{55}$. The Hilbert series of this embedding is given by the formula

$$\frac{(1 + 28q + 273q^2 + 1248q^3 + 3003q^4 + 4004q^5 + 3003q^6 + 1248q^7 + 273q^8 + 28q^9 + q^{10})}{(1-q)^{28}.}$$

Finally, the degree of the homogeneous variety X of dimension $d = 57$ corresponding to the adjoint representation V_λ (of dimension 248) of the exceptional group E_8 is equal to $126937516885200 = 2^4.3^2.5^2.7.31.37.41.43.47.53$.

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The Hilbert Polynomial. Numerical Polynomials. The Multiplicity of a Module over a Minimal Prime Ideal. The Hilbertâ€“Serre Theorem. The Degree of a Projective Variety. The Intersection Theorem and BÃ©zout. An Application of BÃ©zout: Pascal's Theorem.Â In the projective plane, the number of intersection points of two curves equals the product of their degrees, as postulated by BÂ´ezout. This thesis gives a proof of a generalization of this theorem to the intersection of a variety and a hypersurface in higher dimensional projective space applying the notion of the Hilbert polynomial of a graded module to the coordinate ring of the said varieties. 2.3 Expressing the Hilbert polynomial by projective characters Our goal is to express the coefficients of the Hilbert polynomial of V in terms of its projective characters. We first introduce some notation. To any sequence c = (ci)iâˆˆN of elements of a commutative ring such that c0 = 1 and to a partition Î» = (Î»1 , . . . , Î»r) we assign the ring element âˆ†Î» (c) as follows: âˆ†Î» (c) := det ((cÎ»i âˆ'i+j)1â‰¤i,jâ‰¤r) ï£« ï£¶ cÎ»1 cÎ»1 +1 Â·Â·Â· cÎ»1 +râˆ'1 ï£¬ cÎ»2 âˆ'. 1 cÎ»2 Â·Â·Â· cÎ»2 +râˆ'2 ï£· = det ï£¬ , (4) Â· ï£¸ cÎ»r âˆ'r+1 cÎ»r âˆ'r+2 Â·Â·Â· cÎ»r 7 using the convention ci = 0 for i < 0. Note that the value of this deter [3] Hilbert polynomials 293. polynomial of a graded algebra A = k[Xi,..., Xn]/I where k[Xi,..., Xn] is a poly-nomial ring in indeterminates X\,..., Xn over a field k and / is a homogeneous ideal of this ring. Furthermore, by [2, Chapter 9, Theorem 12], the set 7i coincides with the set of all affine Hilbert polynomials in the sense of the following statement (see [2, Chapter 9, Propositions 3 and 4]).Â on polynomials and gives some examples of Hilbert polynomials. THEOREM 2 . 5 . Let f(t) and g(t) be Hilbert polynomials in one variable t, let. m, k â‚¬ N , k > 0, and let c i , . . . , c* be positive integers.Â Setting Go(Â£) = 9P{t) and Gfc(Â£) = GJ_x(t) for A: = 1,2, , we obtain (after a series of routine manipulations with binomial coefficients) that. + I7ft + p+ \ p-4.) \P-4/. A homogeneous polynomial of degree d on $\mathbb P^{mn-1}$ pulls back under s to a polynomial bihomogeneous of degree $(d,d) $ on $\mathbb P^{m-1}\times \mathbb P^{n-1}$. So the required Hilbert polynomial is $$ p_S(d)= \binom {m-1+d}{m-1} \cdot \binom {n-1+d}{n-1} $$. As an application we can calculate the degree of S : it is $dim(S) !=(m+n-2)!$ times the leading coefficient of p_S (that is, the coefficient of d^{m+n-2}) and we find $$ deg(S)=\frac{(m+n-2)!}{(m-1)!(n-1)!} =\binom {m+n-2}{m-1} $.Â That makes it easy to compute the Hilbert series of your ring, since it is the same as that of the quotient by the initial ideal. share | cite | improve this answer |. follow. The Hilbert polynomial of a projective variety V in P*n is defined as the Hilbert polynomial of the homogeneous coordinate ring of V*. Graded algebra and polynomial rings. Polynomial rings and their quotients by homogeneous ideals are typical graded algebras.Â The proof that the Hilbert series has this simple form is obtained by applying recursively the previous formula for the quotient by a non zero divisor (here) and remarking that. Shape of the Hilbert series and dimension. A graded algebra A generated by homogeneous elements of degree 1 has Krull dimension zero if the maximal homogeneous ideal, that is the ideal generated by the homogeneous elements of degree 1, is nilpotent.