Studies in Computational Metaphysics*

— Results of an Interdisciplinary Research Project —

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Abstract

This article summarises and reflects on the results of the research project *"Studies in Computational Metaphysics"* conducted from 2012 to 2017 at Freie Universität Berlin and at Stanford University. The contributions include (i) a theoretical framework towards universal logic reasoning, which utilises classical higher order logic as a universal meta-logic, (ii) the 'implementation' of the approach in higher order interactive and automated theorem provers, (iii) the exemplary application of the approach in selected fields of philosophy, mathematics and computer science, and (iv) the education of a new generation of students to master the approach.

1 Motivation and Preliminary Work

In preliminary research projects, conducted since the mid-nineties, I had gained solid background knowledge regarding the theory and practice of higher order (HO) automated reasoning (see e.g. [119, 19, 38, 18, 9, 23, 24, 8]), regarding HO proof assistants (see e.g. [109, 10, 17, 84, 6, 110, 20]) and regarding their applications in maths, artificial intelligence and education (see e.g. [111, 31, 100, 104, 34, 103]). Moreover, I had developed an initial idea towards the development of a universal logic reasoning framework (see e.g. [28, 29, 11, 30]). The hypothesis has been that shallow semantical embeddings in classical HO logic (HOL)[1] can be used to turn interactive and automated provers for HOL into flexible and powerful reasoning tools for all sorts of quantified and non-quantified non-classical logics and their combinations.

The two main overall objectives of the project *"Studies in Computational Metaphysics (CompMeta)"* therefore have been

1. to further develop the theory of the shallow semantical embeddings approach,
2. to 'implement' the approach in existing, interactive and automated HO reasoning tools,
3. to provide evidence for its universal logic reasoning capabilities in exemplary case studies,
4. and to educate a new generation of students and researchers with the approach.

Regarding (1) and (2), a close collaboration with my parallel projects Leo-II [40] and (later) Leo-III [39, 118] was planned. With respect to (3), a particular emphasis has been on applications in theoretical philosophy, resp. in metaphysics. Why? Since in this area the need for very expressive non-classical logics becomes particularly obvious. For example, hyper-intensional second order modal logic is utilised as the base logic in Zalta's Principia Logico-Metaphysica [127], and similarly expressive logics are discussed and investigated in prominent recent textbooks by Stalnaker and Williamson [115, 123]. Unfortunately, however, there have been very few attempts so far to 'implement' and support such rich logic formalisms in computer systems. With the CompMeta

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[1] HOL has its roots in the logic of Frege's Begriffsschrift [67]. However, the version of HOL as used here is a (simply) typed logic of functions, which has been proposed by Church [61, 5]. It provides lambda-notation, as an elegant and useful means to denote unnamed functions, predicates and sets (by their characteristic functions). Types in HOL eliminate paradoxes and inconsistencies: e.g. the well known Russel paradox (set of sets which do not contains themselves), which can be formalised in Frege's logic, cannot be represented in HOL due to type constraints. More information on HOL and its automation is provided by [25].

project I wanted to address this gap. In more general terms, a core motivation for CompMeta has been to contribute to the pioneering the new area of *computational metaphysics*, which has its roots in the work of Zalta and colleagues at Stanford University [65, 97, 2], and in which many mainstream knowledge representation formalisms in computer science and artificial intelligence (e.g. semantic web taxonomies) fail to deliver due to their lack of expressivity. For the appropriate modeling of *philosophical ontologies* and for the formal analysis of *challenge arguments in philosophy (and beyond)* a suitable expressive modeling and reasoning framework is obviously required. With respect to (3), the interest has been to set-up an interdisciplinary lecture course on computational metaphysics, in which the active use of the CompMeta framework was intended to play a central role.

2 Summary of Project Results

The quest for a most general framework supporting universal reasoning and rational argumentation is very prominently represented in the works of Gottfried Wilhelm Leibniz (1646-1716). He envisioned a *scientia generalis* founded on a *characteristica universalis*, that is, a most universal formal language in which all knowledge (and all arguments) about the world and the sciences can be encoded. This universal logic framework should, so Leibniz, be complemented with a *calculus ratiocinator*, an associated, most general formal calculus in which the truth of sentences expressed in the characteristica universalis should be mechanically assessable by computation.[2]

A quick study of the survey literature on logical formalisms[3] shows that quite the opposite to Leibniz' dream has become reality. Instead of a characteristica universalis, we are today actually facing a very *heterogenous zoo of different logical systems*. The development of these logical systems is typically motivated by e.g. different practical applications, different computational/theoretical properties, different expressivity needs, different schools of origin, etc.

In the course of the CompMeta project the semantical embedding approach has been further developed towards an encompassing universal logic reasoning framework. In addition to its theoretical justification, this approach has a very pragmatic motivation, foremost reuse of tools, simplicity and elegance. It utilises classical higher order logic (HOL) as a unifying *meta-logic* in which (the syntax and semantics) of varying *object logics* can be explicitly modeled and flexibly combined. Off-the-shelf HO interactive and automated theorem provers can then be employed to reason about and within the shallowly embedded logics. This way — so the overall hypothesis of the CompMeta project — Leibniz vision can (at least partially) be realised.

In order to demonstrate and empirically assess the universal reasoning capabilities of this approach, various non-trivial applications studies have been conducted in CompMeta. This includes the rigorous, formal analysis of modern variants of the *ontological argument for the existence of God*, and a related, but more ambitious, formal analysis of Edward Zalta's *Principia Logico-Metaphysica*, which aims at a foundational framework for all of metaphysics, mathematics and the sciences. In both cases, novel insights (e.g. inconsistency results), contributed by automated theorem provers, have been enabled by our approach (see below). These contributions of the CompMeta project had a media repercussion on a global scale and they have led to numerous invited presentations in computer science, mathematics and philosophy.

In addition, the CompMeta project has contributed to the further development of the *theory and practice of automated reasoning for (quantified) non-classical logics*. The project also put a significant amount of resources into the dissemination of results and the outreach to different research communities and to the public. This also included the organisation of scientific events on project related topics. In addition, the project put a strong emphasis on the education of students and scientists with the semantical embedding approach, for example, via the design and implementation of a world-wide novel (and awarded) *lecture course on computational metaphysics* at Freie Universität Berlin. From an epistemological perspective, the project has rebutted the prevalent (mis-)belief, that the automation of HOL is too complicated to be useful in practice. In fact, several non-trivial application studies in metaphysics have been contributed that show quite the opposite, and I claim that these or similar contributions are currently not supported by any other implemented methods and tools in the AI community.

[2]Leibniz characteristica universalis and calculus ratiocinator are prominently discussed in numerous philosophy books and papers. Recommended texts include [91] and [101].

[3]See for example various handbooks on logical formalisms such as [77, 120, 74, 1, 75, 54].

3 Evidence from Application Studies

To assess and demonstrate the coverage, universality and practical relevance of the semantical embedding approach heterogeneous application studies have been conducted in CompMeta. A selection of these application studies is presented next.

Application Study I: Ontological Argument for the Existence of God

Different modern variants of the *Ontological Argument* for the existence of God, one of the still vividly debated masterpiece arguments in metaphysics, have been rigorously analysed on the computer in the course of the CompMeta project.[4] These contributions, most of which were achieved in collaboration with Bruno Woltzenlogel Paleo, received a media repercussion on a global scale.

In the course of the experiments [45, 49, 51], the HO automated theorem prover Leo-II [40] detected a previously unknown inconsistency in Kurt Gödel's prominent, HO modal logic variant [80] of the ontological argument, while Dana Scott's amendment [107] of it was verified for logical soundness in the interactive proof assistants Isabelle/HOL [95] and Coq [52]. Further relevant insights contributed or confirmed by automated theorem provers e.g. include the separation of relevant from irrelevant axioms, the determination of mandatory properties of modalities, and undesired side-implications of the axioms such as the *modal collapse*.[5]

Further variants of Gödel's axioms were proposed by Anderson, Hájek and Bjørdal [4, 3, 81, 82, 83, 53]. These variants have meanwhile also been formally analysed, and automated theorem provers have even contributed to the *clarification of an unsettled philosophical dispute* between Anderson and Hájek [41]. In the course of this work (cf. [46, 14, 47, 13, 43, 48, 44]), *different notions of quantification (actualist and possibilist)* have been utilised and combined within the semantical embedding approach. Moreover, the modal collapse, whose avoidance has been the key motivation for the contributions of Anderson, Bjørdal and Hájek (and many others), has been further investigated [50].

A significant further contribution has been achieved very recently by David Fuenmayor, a philosophy student recruited from in the *computational metaphysics lecture course at Freie Universität Berlin (see below)*. Fuenmayor, in a student project with me (see [72, 117]), formalised the most relevant parts of the textbook *Types, Tableaus, and Gödel's God* [66] by Fitting. In this book Fitting presents another interesting emendation of the ontological argument, which — similar to other recent works — aims at preserving the overall conclusion (necessary existence of God) while at the same time getting rid of the modal collapse. Fitting's means to achieve this is by replacing the foundational logical systems. Instead of an *extensional HO modal logic* he employs a more expressive *intensional HO modal logic*, which enables a different, and as Fitting claims, more adequate interpretation of e.g. the notion of positive poperties in Gödel's argument. Fittings textbook does a very fine job in motivating this adaptation.

Still, a lot remains to be done in this area. Our pilot studies so far only address a small portion of the entire relevant literature on the ontological argument. By extending these studies, I presume, we should be able to identify further issues in human refereed contributions. An encompassing map, that rigorously distinguishes sound from unsound work in this area, would be the ideal outcome of such a follow-up project.

Key insights: Variants of extensional and intensional HO modal logics can easily be 'implemented' in the semantical embedding approach; a very good degree of proof automation can be achieved this way, matching or exceeding the argumentation granularity we typically find in human authored publications on this subject; flexible logic modifications and combinations are supported; the approach is practically highly useful and it combines automated theorem proving with countermodel finding (the latter well supports the detection of typos and minor issues during the formalisation process); as we have demonstrated, the approach is well suited to support a novel, experimental style of computational philosophy.

[4] See e.g. [113] and the references therein for more details on the ontological argument.

[5] The modal collapse [112, 113] is a sort of constricted inconsistency at the level of possible world semantics. The assumption that there may actually be more than one possible world is refuted; this follows from Gödel's axioms as the ATPs quickly confirm. In other words, Gödel's axioms, as a side-effect, imply that everything is determined (we may even say: that there is no free will).

Application Study II: Zalta's Principia Logico-Metaphysica

Formalising and automating masterpiece rational arguments in philosophy with the semantical embedding approach on the computer is not trivial. However, it still leads to comparably small corpora of axioms, lemmata and theorems, and, hence, it does not provide reliable feedback on the scalability of the approach for larger and more ambitious formalisations. For that reason another challenge has been tackled in the CompMeta project: the *Principia Logico-Metaphysica* (PLM) of Zalta [127], which aims at a foundational logical theory for metaphysics, mathematics and the sciences (PLM thus intends to subsume the Principia Mathematica [122]). Zalta has chosen a hyper-intensional, relational second order modal logic S5 as the foundational logic for PLM. It has thus been a challenge question for the CompMeta project, whether this highly non-trivial base logic can still be suitably encoded and automated in the semantical embedding approach. Besides hyper-intensionality, a particular challenge has been to overcome the conceptional gap between the relational core of PLM and the functional core of HOL, and to suitably handle the different strengths of comprehension principles supported in both logics that assert the existence of relations and functions (the use of unrestricted comprehension principles in PLM causes undesirable paradoxes and inconsistencies, cf. [98]). And, of course, a main challenge has also been to deal with the comparably large size of PLM in comparison with the small axiom sets studied in the context of the ontological argument.

The work on the encoding of PLM by utilising the semantical embedding approach started during my stay at Stanford University from Autumn 2015 to Summer 2016. However, my initial attempts to semantically embed PLM's base logic in HOL, following a pure proof theoretic approach, were still unsuccessful. Later, in the course of the computational metaphysics project at Freie Universität Berlin (see below), Zalta then outlined some ideas towards a set theoretical semantics for PLM, which were suggested to him by Peter Aczel. This set theoretic perspective on PLM in turn enabled the development of a suitable shallow semantical embedding of PLM in HOL. It was in fact Daniel Kirchner, a mathematics student recruited from my lecture course on computational metaphysics, who took on the challenge within his masters thesis project at Freie Universität Berlin. Supervised by Zalta and myself, Kirchner has meanwhile succeeded in formalising the entire PLM in Isabelle/HOL by suitably adapting the semantical embedding approach so that it soundly covers the base logic of PLM[86, 87].

Kirchner's work contributes various novel ideas and tools, including the provision of powerful automation means for PLM at different, cross-linked abstraction levels. For example, he developed a direct, tactic-based theorem prover for PLM in Isabelle/HOL, which is reflecting one to one PLM's proof theory. This object-level theorem prover for PLM is connected with the HOL meta-level in Kirchner's work via the shallow semantical embedding he developed, and this link establishes an Isabelle/HOL-internal criterion, modulo expansion of the semantical embedding, for the soundness of his novel prover. Further, similar provers are provided by him at well-defined, intermediate expansion levels. Kirchner's architecture thus provides multiple options for proof automation, ranging from the full expansion of the semantical embedding combined with calls to off-the-shelf reasoning tools integrated with Isabelle/HOL via the Sledgehammer tool [56] to the more intuitive, one to one automation of the proof theory of PLM within Kirchner's new tactic-based theorem prover.

An unexpected, but key result of Kirchner's experimental study has been the discovery of a paradox in PLM [88] (in the spirit of Russel's paradox [92] for Frege's logic of the Begriffsschrift [67]): a deeply-rooted and known paradox is reintroduced in PLM, resp. in the Abstract Object Theory (AOT) underlying PLM, when the logic of complex terms is simply adjoined to AOT's specially-formulated comprehension principle for relations. Kirchner's result constitutes a new and important paradox, given how much expressive and analytic power is contributed by having the two kinds of complex terms in the system. The results also provide a fresh perspective on the question of whether relational type theory or functional type theory better serves as a foundation for logic and metaphysics.

In close collaboration and supported by further experiments with Isabelle/HOL, possible emendations of PLM are currently being studied by Zalta and Kirchner, and the related email exchange between them well illustrates a new dynamics in the scientific discovery process in metaphysics: rigorous experimentation with implementations of foundational logical systems may quicken the scientific discovery process in this area and also foster more reliable results.

Key insights: The semantic embedding approach scales for ambitious and large projects in metaphysics such as PLM; the approach is practically applicable and shows a good, albeit further improvable, degree of automation; with the help of the implemented framework new knowledge has been contributed; moreover, students can be well motivated when using the approach to dive into complex, foundational questions on the edge of current research in metaphysics in short time.

Application Study III: Free Logic and Category Theory

Partiality and undefinedness are prominent challenges in various areas of mathematics and computer science. Unfortunately, however, modern proof assistant systems and automated theorem provers based on traditional classical or intuitionistic logics provide rather inadequate support for these challenge concepts. Free logic [89, 106, 90, 96] offers a theoretically appealing solution, but it has been considered as rather unsuited towards practical utilisation.

In collaboration with Dana Scott, a shallow embedding of free logic in HOL has been developed and implemented in the CompMeta project. Just as for the embeddings mentioned above, various state-of-the-art first-order and HO automated theorem provers and model finders, which are integrated (modulo suitable logic translations) with Isabelle/HOL via the Sledgehammer tool, can now be utilised to automate reasoning in free logic. As a result we obtain an elegant and powerful implementation of an integrated, interactive and automated theorem proving (and model finding) system for free logic.

To demonstrate the practical relevance of our new system, we have developed a series of axioms systems for category theory by generalising the standard axioms for a monoid to a partial composition operation [37, 35, 36]. The purpose of this work has not been to make or claim any contribution to category theory but rather to show how formalisations involving the kind of logic required, in this case free logic, can be implemented and validated within modern proof assistants such as Isabelle/HOL.

We have also addressed the relation of our axiom systems to alternative proposals from the literature, including an axiom set proposed by Freyd and Scedrov in their textbook “Categories, Allegories” [68] for which we have revealed a technical flaw: either all operations, e.g. morphism composition, are total in their work or their axiom system is inconsistent. Thus, in interaction with our implemented framework, we have revealed a minor but still relevant issue in a mathematics textbook that domain experts had missed. The repair for this problem is quite straightforward, however. The solution essentially corresponds to a set of axioms proposed by Scott [108] in the 1970s.

Like in the experiments reported above, our exploration has been significantly supported by series of experiments in which automated reasoning tools have been called from within the proof assistant Isabelle/HOL via the Sledgehammer tool. Moreover, we have obtained very useful feedback at various stages from the model finder Nitpick [55], saving us from making several mistakes.

At the conceptual level our collaboration exemplifies a new style of explorative mathematics which rests on a significant amount of human-machine interaction with integrated interactive-automated theorem proving technology. The experiments we have conducted are such that the required reasoning is often too tedious and time-consuming for humans to be carried out repeatedly with highest level of precision. It is here where cycles of formalization and experimentation efforts in Isabelle/HOL provided significant support. Moreover, the technical inconsistency issue for the axiom system of Freyd and Scedrov was discovered by automated theorem provers, which further emphasises the added value of automated theorem proving in this area.

Key insights: The shallow semantical approach is applicable also to free and inclusive logics, which so far were believed to be too difficult to automate and thus of little practical relevance; quite to the contrary: as our experiments show, the approach is indeed well suited for practical applications, e.g. for the exploration of mathematical theories in domains such as category theory, where partiality and undefinedness play a central role; new knowledge can be discovered this way.

Theoretical Study IV: Universal Cut-Elimination

The development of cut-free calculi for expressive logics, such as quantified non-classical logics, is usually a non-trivial task. However, for a wide range of logics there exists a surprisingly elegant and uniform solution: simply utilise the semantical embedding approach. More precisely, by modeling and studying these logics as semantically embedded fragments of HOL (with Henkin semantics), existing cut-elimination results for HOL may be reused. In the course of the Comp-

Meta project, this idea has been further studied and exemplarily applied for quantified conditional logics [15].

Conditional logics [114, 60], known also as logics of normality or typicality, have many applications, including counterfactual reasoning, default reasoning, deontic reasoning, metaphysical modeling, action planning and reasoning about knowledge. Moreover, it is well known that they fully subsume normal modal logics.[6] In contrast to the rather straightforward, Kripke-style semantics of normal modal logics, conditional logics come e.g. with a HO selection function semantics, which makes them interesting objects of study. While there is broad literature on propositional conditional logics only a few authors have addressed first-order extensions, those include Delgrande [64, 63] and Friedman et al. [69].

The conditional logics studied in the course of the CompMeta project utilise constant- and/or varying-domain first-order quantifiers and they combine these with further quantifiers for propositional variables. Such a rich combination has not been adressed in the literature before. In particular, cut-elimination for the resulting logic(s) was still open (only for propositional conditional logics some related results had been available [99, 102]; cf. also the references therein).

While earlier, practical work in the CompMeta project had already shown that automation of quantified conditional logics is indeed feasible by utilising the semantical embedding approach [12, 22], I switched my attention in the second half of the project to the theoretical challenge of proving cut-elimination [15]. What I showed is that by utilising shallow semantical embeddings of quantified (and non-quantified) conditional logics in HOL, the question whether cut-elimination holds for them can in fact be reduced to proving the faithfulness, resp. soundness and completeness, of the embedding. Proving the faithfulness of the embedding, however, is a much simpler problem than proving cut-elimination directly.

This reduction principle is of course similarly applicable to other object logics in the semantical embedding approach, including many of which cut-elimination is still open. However, special attention has to be payed to cut-simulation, which may render cut-elimination as a pointless criterion (see [15] for further details).

Key insights: Cut-elimination of a given object logic can often be reduced to showing the faithfulness of a shallow semantical embedding of this logic in HOL (with Henkin semantics); the approach has been applied to prove cut-elimination for some variants of quantified conditional logics, for which the question was still open; it should be possible to obtain similar cut-elimination results for many other challenging object logics.

Application Study V: Lecture Course on Computational Metaphysics

The early successes in the CompMeta project inspired the design of a worldwide new lecture course on *computational metaphysics* at Freie Universität Berlin [125, 42]. This lecture course, developed in close collaboration with my PhD students Alexander Steen and Max Wisniewski, received the 2015/16 central teaching award of Freie Universität Berlin.[7] The course received substantial supported from Jasmin Blanchette (Vrije Universiteit Amsterdam), Wolfgang Lenzen (University of Osnabrück), Bruno Woltzenlogel-Paleo (ANU Canberra) and Edward Zalta (Stanford University, USA), who all contributed guest lectures.

Students with heterogenous knowledge backgrounds from computer science, mathematics, philosophy and physics attended the lecture course, and they came from all three major universities of Berlin: Freie Universität Berlin, Technical University Berlin and Humboldt University Berlin. The attendance in the lectures usually varied between 40 and 70 students. 36 students were formally registered for the course and were graded.

The steep learning curves of nearly all students were astonishing, in particular in the second half of the course, when small, heterogeneous student groups were formed to work each on an encoding and formal assessment of a different publication in philosophy or mathematics by adapting and utilising the semantical embedding approach within the Isabelle/HOL proof assistant. The heterogeneous group compositions, the 24/7 feedback from the Isabelle/HOL environment, and the motivating project topics were prime reasons, as we believe, for the steep learning curves and good project results we observed. A selection of project results has meanwhile been presented at conferences and/or published as book chapters or journal articles (see e.g. [72, 7, 73, 71]); further submissions are pending. Several students picked up subsequent projects (see e.g. the

[6] The box operator can be defined in terms of the more expressive conditional operator.

[7] http://www.fu-berlin.de/campusleben/lernen-und-lehren/2016/160428-lehrpreis/index.html

projects by Fuenmayor [72, 117, 12] and Kirchner [87, 86] as mentioned above) and turned them into Bachelor's or Mather's thesis projects [93, 70, 78, 85, 105].

A key factor in the successful implementation of the course has been, that a single methodology and overall technique (the semantical embedding approach) was used throughout, enabling the students to quickly adopt a wide range of different logic variants in short time within a single proof assistant framework (Isabelle/HOL). The interdisciplinary course concept appears well suited to foster a much improved logic education across disciplines.

Key insights: The semantical embedding approach is suited to support a novel form of university level logic education at to heterogeneous groups of students; excellent learning curves are possible; new teaching methods are enabled in interaction with HO proof assistants and automated theorem provers.

4 Further Results and Comments

A range of related application studies (contributed partly also by collaborators) has not been mentioned above. Amongst others, these works include semantical embeddings of multivalued logic SIXTEEN [116], nominal logics [124], temporal logics [62], paraconsistent logics, intuitionistic modal logics, etc. I also did not report on the work of Daniel Streit (another student recruited from the computational metaphysics lecture course) on the formalisation of Boolos' textbook on provability logic [58].

A relevant and challenging future application direction of the semantical embedding approach lies in the modeling of legal, ethical, social and cultural norms in intelligent machines [16]. To enable such applications, I am currently, in a collaboration with van der Torre and Parent from University of Luxemburg, adapting the semantical embedding approach to cover recent developments in the area of deontic logics [76]. Standard deontic logic, which is just a normal modal logic, is obviously already covered by the approach. More challenging is the semantical embedding of e.g. input/output logic [94], dyadic logic [59]. First experiments in this direction are promising [21].

A Note on Invention and Creativity in Automated Theorem Provers. As reported above, the theorem prover Leo-II detected the inconsistency of the axioms in Gödel's original variant of the ontological argument, and, as far as I am aware, this inconsistency was not known to philosophers before. The clue in the proof of falsity from the axioms, as is described in more detail in [51, 12], is the *empty essence lemma*: from Gödel's [80] definition of essential properties (essence) it follows that the empty property (i.e., the everywhere false property) is an essential property of every individual. Dana Scott [107] slightly modified Gödel's definition of essence in his variant of the ontological argument (for cosmetic reasons — the inconsistency was not known to him at the time), with the effect that the empty essence lemma is no longer valid.

In its successful, automatic discovery of the inconsistency, the Leo-II prover had to guess the instantiation of the empty property for a second-order variable during proof search. This part in the proof is non-analytic: inspection shows that one cannot synthesise this particular instantiation by e.g. unification with existing information (terms) in the search space. In fact, blind guessing of this instantiation seems unavoidable here. I consider this as a small but nevertheless very interesting example for a true discovery (based on guessing and checking) by an automated automated theorem prover; and this discovery has been philosophically relevant.

Improved Infrastructure for HO Interactive and Automated Reasoning. In the course of the above works, and in close collaboration with the Leo-III project, the CompMeta project fostered the development of a reusable theorem proving infrastructure for a range of non-classical logics [79, 57, 126, 33, 26, 32, 27]. This includes various, reusable encodings in Isabelle/HOL syntax, Coq syntax and in TPTP syntax, which have been all made publicly available.[8] Moreover, this includes a flexible pre-processing module [39, 118] for the Leo-III prover (and any other TPTP THF [119] compliant prover). This preprocessor turns Leo-III into a flexible reasoner for a very wide range of propositional and quantified modal logics. In fact, no other implemented system is available today which covers a wider range of modal logic variants than Leo-III in combination

[8] See https://github.com/FormalTheology/GoedelGod; various other links have been given before.

with this tool, and this approach can easily be extended for many other non-classical logics that have been mentioned above.

5 Conclusion

The CompMeta project has further developed, implemented and applied the (shallow) semantical embedding approach, which utilises HOL at meta-level to encode (combinations of) a wide range of non-classical logics. This approach to universal logical reasoning has many relevant applications in artificial intelligence, computer science, philosophy, mathematics and natural language processing. Automation of reasoning is achieved indirectly with off-the-shelf reasoning tools as currently developed, integrated and deployed in modern HO proof assistants. The range of possible further applications of the approach is far reaching, and — as has been evidenced in this article — even scales for non-trivial rational arguments, including but not limited to masterpiece arguments in philosophy. A relevant and challenging future application direction concerns the application of the semantical embedding approach for the modeling of ethical, legal, social and cultural norms in intelligent machines, ideally in combination with the realisation of human-intuitive forms of rational arguments in machines complementing internal decision making means at the level of statistical information and subsymbolic representations. To enable such applications, suitable adaptations of the semantical embedding approach to cover recent developments in the area of deontic logics are required.

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Computational metaphysics, as we practice it, is the implementation and investigation of formal, axiomatic metaphysics (i.e., the study of metaphysics using formally represented axioms and premises to derive conclusions) in an automated reasoning environment. In one of the major strands of our research, we've worked within the axiomatic theory of abstract objects developed at the Metaphysics Research Lab at Stanford University. We have represented some of the axioms and definitions of abstract object theory in the syntax of various automated reasoning systems. Computational metaphysics is computational philosophy with a focus on metaphysics. In this paper, we (a) develop results in modal metaphysics whose discovery was computer assisted, and (b) conclude that these results work not only to the obvious bene-t of philosophy but also, less obviously, to the benet of computer science, since the new computational techniques that led to these results may be more broadly applicable within computer science.Â Initial studies10 investi-gated GÂ¨odelâ€™s and Scottâ€™s variants of the argument within the higher-order automated theorem prover What Is Metaphysics? Metaphysics acknowledges and respects the beauty in ALL of Godâ€™s Creation. Metaphysics is religion without dogma. Metaphysics does not explore religious beliefs and laws created by man, but rather, it explores the immutable laws of nature, set by The Creator, God/Universal Presence, in the creation of the Universe. Metaphysics is a branch of philosophy that studies the ultimate nature of existence, reality, and experience without being bound to any one theological doctrine or dogma. Metaphysics includes all religions but transcends them all. Metaphysics is the study of ult Metaphysics is the most abstract branch of philosophy. Itâ€™s the branch that deals with the â€œfirst principlesâ€ of existence, seeking to define basic concepts like existence, being, causality, substance, time, and space. Within metaphysics, one of the main sub-branches is ontology, or the study of being. These two terms are so closely related that you can often hear people use â€œmetaphysicsâ€ and â€œontologyâ€ interchangeably.Â However, the two concepts are not exactly the same: whereas metaphysics studies the general nature of reality, ontology specifically studies the idea of being. Another way to put this would be to say that ontology asks â€œwhatâ€ while metaphysics asks â€œhow,â€ although this is only a generalization. II. Metaphysics vs. Epistemology.