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The convexity of solution of a class Hessian equation in bounded convex domain in $\mathbb{R}^3$

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Abstract

We use the deformation methods to obtain the convexity of solution of a class Hessian equation in bounded convex domain in $\mathbb{R}^3$.

Keywords: Nonlinear elliptic equations; Convexity of solutions

1. Introduction

The convexity is an issue of interest for a long time in partial differential equations, it is intimately related to the study of geometric properties of solutions of general elliptic partial differential equations. It was Gabriel [8] first obtained that the level sets of the Green function in three-dimension convex domains in $\mathbb{R}^3$ are strictly convex. Makar-Limanov [18] considered the following elliptic boundary value problem:

$$\begin{aligned} \Delta u &= -1 \quad \text{in } \Omega, \\ u &= 0 \quad\ \ \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

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in bounded plane convex domain Ω. By an ingenious argument involving the maximum principle, he proved that $u^{1/2}$ is strictly concave.

In 1976, Brascamp, Lieb [3] established the log-concavity of the fundamental solution of diffusion equation with convex potential. As a consequence, they proved the log-concavity of the first eigenfunction of Laplace operator in convex domains.

For the case of dimension two, Acker, Payne, Philippin [1] utilized the idea of Makar-Limanov [18] to obtain a new proof for the Brascamp–Lieb's result. Along the idea in [1,18], Ma [17] gave a new proof of problem (1.1), and he obtained an optimal lower bound of the Gaussian curvature for the graph of $u^{1/2}$.

In 1983 Korevaar [14] introduced a very useful technique now named Korevaar's concavity maximum principle, and established convexity results for the mean curvature type equations under certain boundary value conditions. Then immediately new proofs of the log-concavity of the first eigenfunction of convex domains was given respectively by Korevaar [15] and Caffarelli, Spruck [5]. In different extent, Kawohl [12] (for the intermediate case) and Kennington [13] improved Korevaar's maximum principle, which enabled them to give a higher-dimensional generalization of the result of Makar-Limanov [18]. In particular, Kennington pointed out that the concavity number $\frac{1}{2}$ of u is sharp in Eq. (1.1) in higher-dimension case.

But Korevaar's maximum principle have strong restrictions in many applications, for example we cannot obtain the Gabriel [8] results. In a fundamental work of Singer, Wong, Yau, Yau [20] and Caffarelli, Friedman [4], they devised a new deformation technique to deal with the convexity. Caffarelli, Friedman [4] established the strict convexity of level sets of solution of some equations in two-dimensional convex domain, especially they got the strict log-concavity of the first eigenfunction of Laplace operator in plane convex domains. Korevaar, Lewis [16] generalized the deformation method to higher dimensions, and obtained the strict concavity of $u^{1/2}$ in Eq. (1.1) in higher-dimension case.

Recently, Alvarez, Lasry, Lions [2] generalized the approach of Korevaar [14] and Kennington [13] to a large class fully nonlinear second order elliptic equations:

$$F\big(x, u(x), Du(x), D^2u(x)\big) = 0 \tag{1.2}$$

in convex domain Ω in $\mathbb{R}^n$. But the method cannot give the strict convexity of the solutions. Naturally one wish to generalize the deformation method of Caffarelli, Friedman [4] and Korevaar, Lewis [16] to fully nonlinear version. Motivated by some differential geometry problems, such deformation lemma (constant rank theorem) was established in Guan, Ma [9] and Caffarelli, Guan, Ma [7], and they concluded the general convexity principle for the following elliptic equations:

$$F\big(D^2u(x)\big) = f\big(x, u(x), Du(x)\big). \tag{1.3}$$

They found the structure condition on $F(A)$ just the case as in Alvarez, Lasry, Lions [2], that is $-F(A^{-1})$ is concave on A.

The more detail history and results on the convexity of solutions of elliptic partial differential equations please consult the book by Kawohl [11] and the survey paper by Guan, Ma [10].

In this paper we shall generalized the results of Makar-Limanov [18] and Korevaar, Lewis [16] on Eq. (1.1) to a class Hessian equation in three-dimension case. First we need some preparation to state our theorem.

Let S_k be the kth elementary symmetric function, that is for $1 \leqslant k \leqslant n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$,

$$S_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}. \tag{1.4}$$

In a seminal paper by Caffarelli, Nirenberg, Spruck [6], they considered the following Dirichlet problem for Hessian equation

$$\begin{aligned} &S_k\big(\lambda\big\{D^2u\big\}\big) = f(x) > 0 \quad && \text{in } \Omega \subset \mathbb{R}^n, \\ &u = \phi && \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

where $2 \leqslant k \leqslant n-1$ and $\lambda\{D^2u\}$ means the eigenvalues of Hessian matrix $\{u_{ij}(x)\}$, Ω is a smooth bounded domain in $\mathbb{R}^n$.

In order to state their theorem, we first give some notations from [6].

Definition 1. (See [6].) For $1 \leqslant k \leqslant n$, define

$$\Gamma_k = \big\{\lambda \in \mathbb{R}^n \colon\ S_1(\lambda) > 0, \dots, S_k(\lambda) > 0\big\}.$$

A function $u \in C^2(\Omega)$ is called an admissible solution of (1.5) if the eigenvalues of $\{u_{ij}(x)\}$ belong to Γ_k for each $x \in \Omega$.

In order to solve the Dirichlet boundary value problem (1.5), they [6] found that the following necessary condition for the smooth bounded domain $\Omega \subset \mathbb{R}^n$. If we let $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ be the principal curvature of the boundary $\partial\Omega$, then $\kappa \in \Gamma_{k-1}$.

Now let us state their existence theorem on the admissible solutions for Eq. (1.5).

Theorem 1. *(See [6].) If $f(x) \in C^\infty(\overline{\Omega})$, $f(x) > 0$ on $\overline{\Omega}$, $\phi \in C^\infty(\partial\Omega)$, Ω is a smooth bounded domain in $\mathbb{R}^n$ with principal curvature $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ of $\partial\Omega$ satisfies $\kappa \in \Gamma_{k-1}$. Then for the Dirichlet boundary value problem* (1.5) *there exists a unique admissible solution $u(x) \in C^\infty(\overline{\Omega})$.*

A natural question is whether the solution obtained by [6] has some similar convexity as in Laplace equation case (1.1) (see for example [18] and [16]).

In this work we answer this question for the following simplest case. We consider the following equation in $\mathbb{R}^3$:

$$\begin{aligned} &S_2\big(\lambda\big\{D^2u\big\}\big) = 1 \quad && \text{in } \Omega \subset \mathbb{R}^3, \\ &u = 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.6}$$

The following theorem is our main result.

Theorem 2. *Suppose $u \in C^\infty(\overline{\Omega})$ is the admissible solution of* (1.6)*, and Ω is a strictly convex smooth bounded domain in $\mathbb{R}^3$, then $v := -(-u)^{1/2}$ is strictly convex, and the convexity index $\frac{1}{2}$ is sharp.*

Remark 1. In the above theorem we only obtain the three-dimension case, it seems that for the higher dimension we need other methods to get the similar results. The another question is how about the higher-order elementary symmetric function. We believe the similar results holds.

Remark 2. Also in three-dimension case, we can use the above calculation to get the generalization of the theorem by Brascamp, Lieb [3], then we obtain the Brunn–Minkowski inequality for the eigenvalue of a class Hessian operator and prescribing the equality case. This is a joint work with Professor Liu Pan.

The plan of the paper is as follows. In Section 2, we prove the Hessian of v has constant rank if the function v in Theorem 2 is convex. In Section 3, we show v is strictly convex by continuity method and the index $\frac{1}{2}$ is sharp.

2. Constant rank theorem

If we let $v = -(-u)^{1/2}$, then Eq. (1.6) is equivalent to

$$F\big(v, Dv, D^2v\big) = \frac{1}{4}S_2(u_{ij}) = \frac{1}{4} \qquad \text{in } \Omega \subset \mathbb{R}^3, \tag{2.1}$$

$$v = 0 \qquad \text{on } \partial\Omega, \tag{2.2}$$

where

$$F\big(v, Dv, D^2v\big) = v^2 S_2(v_{ij}) + v\big(v_2^2 + v_3^2\big)v_{11} + v\big(v_1^2 + v_3^2\big)v_{22} + v\big(v_1^2 + v_2^2\big)v_{33} - vv_1v_2(v_{12} + v_{21}) - vv_1v_3(v_{13} + v_{31}) - vv_2v_3(v_{23} + v_{32}). \tag{2.3}$$

Lemma 1 *(Constant rank theorem). Let $u \in C^4(\Omega)$ is an admissible solution of Eq.* (1.6)*, where $\Omega \subset \mathbb{R}^3$ is any domain. If $v := -(-u)^{1/2}$ is a convex function, i.e. the Hessian matrix of v is semipositive in Ω, i.e. $W := \{v_{ij}\} \geqslant 0$, then (v_{ij}) has constant rank in Ω.*

Proof. For $n = 3$, the rank of matrix $\{v_{ij}\}$ can only be in three cases: 1, 2 or 3. Rank is equal to 1 is impossible, since rank 1 implies $S_2(v_{ij})$ degenerate, which contradicts to the condition that u is an admissible solution of Eq. (1.6). Now we suppose W attain to the minimal rank 2 at some point $z_0 \in \Omega$, we will prove that the rank of W always be 2 in Ω, otherwise the rank of W is equal to 3 in Ω.

We shall use the strong minimum principle to prove the lemma. Let

$$P(x) = \det v_{ij}(x),$$

and $P(z_0) = 0$. We shall show that there exists an open small neighborhood O of z_0, such that $P(x) \equiv 0$ in O. If it is true, it implies the set $\{x \mid P(x) = 0\}$ is an open set. But it is also closed, then we get $P(x) \equiv 0$ in Ω since Ω connected, i.e. W is of constant rank 2.

In the following proof, we us the notations in [4] and [16]. For two functions defined in the open set $O \subset \Omega$, $y \in O$, we say that $h(y) \lesssim k(y)$ is provided if there exist positive constants c_1 and c_2 such that

$$(h - k)(y) \leqslant \big(c_1|\nabla P| + c_2 P\big)(y). \tag{2.4}$$

We also write $h(y) \sim k(y)$ if $h(y) \lesssim k(y)$ and $k(y) \lesssim h(y)$. Next, we write $h \lesssim k$ if the above inequality holds in O, with the constants c_1, and c_2 independent of y in this neighborhood. Finally, $h \sim k$ if $h \lesssim k$ and $k \lesssim h$.

We shall show that

$$\sum_{i,j=1}^{3} F^{ij} P_{ij} \lesssim 0, \tag{2.5}$$

in an open small neighborhood O of z_0.

Since $P \geqslant 0$ in Ω and $P(z_0) = 0$, then it follows from the strong minimum principle that $P(z) \equiv 0$ in O. In order to prove (2.5) at an arbitrary point $z \in O$, we choice the normal coordinates, i.e. we perform a rotation T_z about z so that in the new coordinates W is diagonal at z, and $v_{11} \geqslant v_{22} \geqslant v_{33}$ at z. Consequently we can choice T_z to vary smoothly with z. If we can establish (2.5) at z under the assumption that W is diagonal at z, then going back to the original coordinates we find that (2.5) remain valid with new coefficients c_1, c_2 in (2.4), depending smoothly on the independent variable. Thus it remains to establish (2.5) under the assumption that W is diagonal at z.

For rank is at least 2, then there exists a positive constant C, which depends only on $\|v\|_{C^4}$, such that $v_{11} \geqslant v_{22} \geqslant C$ at z. In the following, all calculations are at the point z using the relation "$\lesssim$", with the understanding that the constants in (2.5) are under control.

Next we compute P and its first and second derivatives in the directions x_i, x_j. Since W is diagonalized at z then

$$0 \sim P \sim v_{33}, \qquad 0 \sim P_i \sim v_{33i}, \tag{2.6}$$

$$P_{ij} \sim v_{11}v_{22}v_{33ij} - 2v_{11}v_{23i}v_{23j} - 2v_{22}v_{13i}v_{13j}. \tag{2.7}$$

The following are some notations we will use later:

$$F^{ij} = \frac{\partial F}{\partial v_{ij}}, \qquad F_{p_l} = \frac{\partial F}{\partial v_l}, \qquad F_v = \frac{\partial F}{\partial v},$$

$$F^{ij,rs} = \frac{\partial^2 F}{\partial v_{ij}\partial v_{rs}}, \qquad F^{ij}_{p_l} = \frac{\partial^2 F}{\partial v_{ij}\partial v_l}, \qquad F^{ij}_v = \frac{\partial^2 F}{\partial v_{ij}\partial v},$$

$$F_{p_k,p_l} = \frac{\partial^2 F}{\partial v_k\partial v_l}, \qquad F_{p_k,v} = \frac{\partial^2 F}{\partial v_k\partial v}, \qquad F_{vv} = \frac{\partial^2 F}{\partial v^2}.$$

By calculations we get:

$$\begin{aligned}
F &\sim v^2 v_{11}v_{22} + v v_{11}\big(v_2^2 + v_3^2\big) + v v_{22}\big(v_1^2 + v_3^2\big),\\
F^{ij} &= \frac{\partial F}{\partial u_{rs}}\frac{\partial u_{rs}}{\partial v_{ij}} = -2v\frac{\partial F}{\partial u_{ij}},\\
F^{11} &\sim v^2 v_{22} + v\big(v_2^2 + v_3^2\big),\\
F^{22} &\sim v^2 v_{11} + v\big(v_1^2 + v_3^2\big),\\
F^{12} &= F^{21} \sim -v v_1 v_2,
\end{aligned} \tag{2.8}$$

$$
\begin{aligned}
F_v &= \frac{\partial F}{\partial u_{rs}}\frac{\partial u_{rs}}{\partial v} = -2v_{ij}\frac{\partial F}{\partial u_{ij}} \\
&\sim 2vv_{11}v_{22} + v_{22}\left(v_1^2 + v_3^2\right) + v_{11}\left(v_2^2 + v_3^2\right) \\
&\sim vv_{11}v_{22} + \frac{F}{v} \sim vS_2(v_{ij}) + \frac{1}{4v} \\
&\sim \frac{F^{11}v_{11} + F^{22}v_{22}}{v}.
\end{aligned} \tag{2.9}
$$

Note $v \neq 0$ in the above relations because the original equation and boundary condition tell us $v < 0$ in Ω. Furthermore,

$$
\begin{aligned}
&F_{vv} = 2S_2(v_{ij}) \sim 2v_{11}v_{22}, \\
&F_v^{ij} = -\frac{1}{2}\frac{\partial S_2}{\partial u_{ij}} + vv_{rs}\frac{\partial^2 S_2}{\partial u_{ij}u_{rs}}, \\
&F_v^{11} \sim 2vv_{22} + v_2^2 + v_3^2 \sim \frac{F^{11}}{v} + vv_{22}, \\
&F_v^{22} \sim 2vv_{11} + v_1^2 + v_3^2 \sim \frac{F^{22}}{v} + vv_{11}, \\
&F_v^{12} \sim F_v^{21} \sim -v_1v_2 \sim \frac{F^{12}}{v} \sim \frac{F^{21}}{v}, \\
&F^{11,22} = F^{22,11} = v^2, \\
&F^{12,21} = F^{21,12} = -v^2.
\end{aligned} \tag{2.10}
$$

Differentiate (2.1) once in x_3 to get

$$
F^{ij}v_{ij3} + F_{p_l}v_{l3} + F_v v_3 = 0. \tag{2.11}
$$

In fact at z it just be

$$
F^{11}v_{113} + F^{22}v_{223} + 2F^{12}v_{123} = -F_v v_3. \tag{2.12}
$$

Differentiate (2.1) along the direction of x_3 once more. We obtain

$$
\begin{aligned}
&F^{ij}v_{ij33} + F^{ij,rs}v_{ij3}v_{rs3} + 2F_{p_l}^{ij}v_{ij3}v_{l3} + 2F_v^{ij}v_{ij3}v_3 \\
&\quad + F_{p_l}v_{l33} + F_{p_lp_s}v_{l3}v_{s3} + 2F_{p_lv}v_{l3}v_3 + F_v v_{33} + F_{vv}v_3^2 = 0.
\end{aligned} \tag{2.13}
$$

Note where and thereafter the repeated indices means the sum of these terms. Using (2.6) and the fact (v_{ij}) is diagonal, one may see

$$
F^{ij}v_{ij33} \sim -F^{ij,rs}v_{ij3}v_{rs3} - 2F_v^{ij}v_{ij3}v_3 - F_{vv}v_3^2.
$$

With (2.7) then

$$\begin{aligned}
\frac{F^{ij}P_{ij}}{v_{11}v_{22}} &\sim -F^{ij,rs}v_{ij3}v_{rs3}-2F_v^{ij}v_{ij3}v_3-F_{vv}v_3^2\\
&\quad-\frac{2}{v_{11}}F^{ij}v_{13i}v_{13j}-\frac{2}{v_{22}}F^{ij}v_{23i}v_{23j}\\
&\sim -2v^2v_{113}v_{223}+2v^2v_{123}^2\\
&\quad-2\big(F_v^{11}v_{113}+F_v^{22}v_{223}+2F_v^{12}v_{123}\big)\frac{F^{11}v_{113}+F^{22}v_{223}+2F^{12}v_{123}}{-F_v}\\
&\quad-2S_2(v_{ij})\left(\frac{F^{11}v_{113}+F^{22}v_{223}+2F^{12}v_{123}}{-F_v}\right)^2\\
&\quad-\frac{2}{v_{11}}\big(F^{11}v_{113}^2+F^{22}v_{123}^2+2F^{12}v_{113}v_{123}\big)\\
&\quad-\frac{2}{v_{22}}\big(F^{11}v_{123}^2+F^{22}v_{223}^2+2F^{12}v_{223}v_{123}\big),
\end{aligned}\tag{2.14}$$

where we used (2.12), and from (2.9) we know $F_v\neq 0$. Multiplying both the sides of the above relation by F_v^2, one can write out the coefficients of each term in the right-hand side as follows:

$$\begin{aligned}
v_{123}^2\colon\quad & 2v^2F_v^2+8F_vF_v^{12}F^{12}-8S_2(v_{ij})\big(F^{12}\big)^2-\frac{2F^{22}}{v_{11}}F_v^2-\frac{2F^{11}}{v_{22}}F_v^2\\
&\sim 2\left(v^2-\frac{F^{22}v_{22}+F^{11}v_{11}}{v_{11}v_{22}}\right)F_v^2+8\left(vS_2(v_{ij})+\frac{1}{4v}\right)\frac{F^{12}}{v}F^{12}-8S_2(v_{ij})\big(F^{12}\big)^2\\
&\sim \frac{2(F^{12})^2}{v^2}-\frac{F_v^2}{2v_{11}v_{22}},
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
v_{113}^2\colon\quad & 2F_vF_v^{11}F^{11}-2S_2(v_{ij})\big(F^{11}\big)^2-2\frac{F^{11}}{v_11}F_v^2\\
&\sim 2F_v\left(\frac{F^{11}}{v}+vv_{22}\right)F^{11}-2S_2(v_{ij})\big(F^{11}\big)^2-\frac{2F^{11}F_v}{v_{11}}\left(vv_{11}v_{22}+\frac{1}{4v}\right)\\
&\sim \frac{(F^{11})^2}{2v^2}-\frac{F^{11}F_v}{2vv_{11}}\\
&\sim -\frac{F^{11}F^{22}v_{22}}{2v^2v_{11}},
\end{aligned}\tag{2.16}$$

$$\begin{aligned}
v_{123}v_{113}\colon\quad & 2F_v\big(2F_v^{11}F^{12}+2F^{11}F_v^{12}\big)-8S_2(v_{ij})F^{11}F^{12}-\frac{4F^{12}}{v_{11}}F_v^2\\
&\sim 4F_v\left(\left(\frac{F^{11}}{v}+vv_{22}\right)F^{12}+\frac{F^{11}F^{12}}{v}\right)\\
&\quad-8S_2(v_{ij})F^{11}F^{12}-4F^{12}F_v\frac{vv_{11}v_{22}+\frac{1}{4v}}{v_{11}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2F^{11}F^{12}}{v^2} - \frac{F^{12}F_v}{vv_{11}} \\
&= \frac{F^{12}}{v^2 v_{11}}\big(F^{11}v_{11} - F^{22}v_{22}\big), \qquad (2.17)
\end{aligned}
$$

$$
\begin{aligned}
v_{113}v_{223}\colon \quad & -2v^2F_v^2 + 2F_v\big(F_v^{11}F^{22} + F_v^{22}F^{11}\big) - 4S_2(v_{ij})F^{11}F^{22} \\
&= \frac{F^{11}F^{22}}{v^2}. \qquad (2.18)
\end{aligned}
$$

For the symmetry of sub-indexes 1 and 2, we also get:

$$
\begin{aligned}
v_{223}^2\colon \quad & 2F_vF_v^{22}F^{22} - 2S_2(v_{ij})\big(F^{22}\big)^2 - 2\frac{F^{22}}{v_22}F_v^2 \\
&\sim -\frac{F^{11}F^{22}v_{11}}{2v^2v_{22}}, \qquad (2.19)
\end{aligned}
$$

$$
v_{123}v_{223}\colon \quad \frac{F^{12}}{v^2v_{22}}\big(F^{22}v_{22} - F^{11}v_{11}\big).
$$

So at last

$$
\begin{aligned}
\frac{F^{ij}P_{ij}}{v_{11}v_{22}}F_v^2 \sim -\Bigg(&\sqrt{\frac{F^{11}F^{22}v_{22}}{2v^2v_{11}}}v_{113} - \sqrt{\frac{F^{11}F^{22}v_{11}}{2v^2v_{22}}}v_{223} \\
&- \frac{F^{12}(F^{11}v_{11} - F^{22}v_{22})}{\sqrt{2v^2F^{11}v_{11}F^{22}v_{22}}}v_{123}\Bigg)^2 - Av_{123}^2, \qquad (2.20)
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -\frac{(F^{12})^2(F^{11}v_{11} - F^{22}v_{22})^2}{2v^2F^{11}v_{11}F^{22}v_{22}} - \frac{2(F^{12})^2}{v^2} + \frac{F_v^2}{2v_{11}v_{22}} \\
&\sim \frac{-(F^{12})^2[(F^{11}v_{11})^2 + (F^{22}v_{22})^2 - 2F^{11}v_{11}F^{22}v_{22}]}{2v^2F^{11}v_{11}F^{22}v_{22}} \\
&\quad - \frac{2(F^{12})^2}{v^2} + \frac{(F^{11}v_{11} + F^{22}v_{22})^2}{2v^2v_{11}v_{22}} \\
&\sim \left(\frac{F^{11}v_{11}}{2v^2F^{22}v_{22}} + \frac{F^{22}v_{22}}{2v^2F^{11}v_{11}} + \frac{1}{v^2}\right)\big[F^{11}F^{22} - \big(F^{12}\big)^2\big]. \qquad (2.21)
\end{aligned}
$$

It is obvious that

$$
\left(\frac{F^{11}v_{11}}{2v^2F^{22}v_{22}} + \frac{F^{22}v_{22}}{2v^2F^{11}v_{11}} + \frac{1}{v^2}\right) > 0.
$$

Moreover,

$$
\begin{aligned}
F^{11}F^{22} - \left(F^{12}\right)^2 &\sim \left[v^2 v_{22} + v\left(v_2^2 + v_3^2\right)\right]\left[v^2 v_{11} + v\left(v_1^2 + v_3^2\right)\right] - v^2 v_1^2 v_2^2 \\
&\sim F v^2 + v^2\left(v_1^2 v_3^2 + v_2^2 v_3^2 + v_3^4\right) \\
&\gtrsim 0. \qquad (2.22)
\end{aligned}
$$

It follows that $A \gtrsim 0$, and the quantity in (2.20) $\lesssim 0$, then (2.5) holds. □

3. Proof of Theorem 2

Now we begin the proof of Theorem 2. After we have Lemma 1 (constant rank theorem), it is well known the proof of Theorem 2 is standard, see for example the papers by Caffarelli, Friedman [4] and Korevaar, Lewis [16].

First we have the boundary convexity estimates for the function $v := -(-u)^{1/2}$, where the function $u(x)$ is the admissible solution of Eq. (1.6), which follows from the following proposition if we take $f(t) = -(-t)^{1/2}$.

Proposition 1. *(See e.g. [5,15].) Let $\Omega \subset \mathbb{R}^n$ be smooth, bounded and strictly convex (i.e. all the principal curvature of $\partial\Omega$ are positive). Let $u \in C^2(\overline{\Omega})$ satisfy*

$$u < 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{and} \quad Du \cdot \nu > 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

where ν is the exterior normal to $\partial\Omega$. Let

$$\Omega_\varepsilon = \left\{x \in \Omega\colon\ d(x, \partial\Omega) > \varepsilon\right\} \tag{3.2}$$

and let $v = f(u)$. Then for small enough $\varepsilon > 0$ the function v is strictly convex in a boundary strip $\Omega \setminus \Omega_\varepsilon$ if f satisfies

$$\text{(i)} \quad f' > 0, \qquad \text{(ii)} \quad f'' > 0, \qquad \text{(iii)} \quad \lim_{u \to 0^-} \frac{f'}{f''} = 0. \tag{3.3}$$

Now we use the deformation technique combine with Lemma 1 (constant rank theorem) to obtain the proof of Theorem 2 as in Korevaar, Lewis [16], for completeness we repeat partly their proof.

Proof of Theorem 2. Let us illustrate the continuity method to end Theorem 2. Now, if Ω is the unit ball B, then the solution of (2.1), (2.2) is

$$v(x) = -\left[\left(1 - |x|^2\right)/\sqrt{2n(n-1)}\,\right]^{1/2}, \quad x \in B.$$

So clearly v is strictly convex. For an arbitrary strictly convex domain Ω, set $\Omega_t = (1-t)B + t\Omega$, $0 \leqslant t \leqslant 1$. Then from the theory of convex bodies (see for example Sections 1.7, 1.8 and 2.5 in the book [19], and Section 3.1 in the book [21]) we can deform B continuously into Ω by the family (Ω_t), $0 \leqslant t < 1$, of strictly convex domain in such a way that $\partial\Omega_t \to \partial\Omega_s$ as $t \to s$ in the sense of Hausdorff distance, whenever $0 \leqslant s \leqslant 1$. And the deformation also is chosen so that $\partial\Omega_t$, $0 \leqslant t < 1$, can be locally represented for some α, $0 < \alpha < 1$, by a function whose norm in the space $C^{2,\alpha}$ of functions with Hölder continuous second derivatives depends only on δ, whenever $0 < t \leqslant \delta < 1$.

Suppose $u \in C^\infty(\overline{\Omega})$ is the admissible solution of (1.6), $v := -(-u)^{1/2}$ and H_t is the corresponding Hessian matrix of v. First H_0 is positive definite, and from the boundary estimates (Proposition 1) we have H_δ is positive definite in an ε neighborhood of $\partial\Omega_\delta$. From the $C^{2,\alpha}$ estimates of the solution u on the Hessian equation [6], we know this bounded depends only the uniformly bounded geometry of Ω_t which depends the geometry Ω and t. We conclude that if $v(.,s)$ is strictly convex for all $0 \leqslant s < t$, then $v(.,t)$ is convex.

So if for some δ, $0 < \delta < 1$, H_δ is positive semi-definite but not positive definite in Ω_δ, we say it is impossible by constant rank theorem (Lemma 1) and boundary estimates (Proposition 1). We conclude H_δ is positive definite. Then $v = -\sqrt{-u}$ is strictly convex in Ω. □

Remark 3. The convexity index $\frac{1}{2}$ in Theorem 2 is sharp, we give an analogous counterexample in [13].

Lemma 2. *Assume Ω is a convex domain in $\mathbb{R}^n$, and $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$, $u|_\Omega < 0$. If for $\alpha > 0$, $y \in \partial\Omega$, $z \in \Omega$, $\limsup_{t\to 0^+} t^{-1/\alpha} u((1-t)y + tz) = 0$, then $-u$ is not α-concave in Ω.*

Proof. Suppose $-u$ is α-concave. Then for $t \in (0,1)$, the concavity of $(-u)^\alpha$ implies that

$$(-u)^\alpha\big((1-t)y + tz\big) \geqslant (1-t)(-u)^\alpha(y) + t(-u)^\alpha(z) = t(-u)^\alpha(z),$$

so that $t^{-1/\alpha} u((1-t)y + tz) \leqslant u(z) < 0$, which contradicts assumption. □

Now we show the index $\frac{1}{2}$ is sharp in Theorem 2.

Proof of the sharpness of index $\frac{1}{2}$. Let $n = 3$ and $x \in \mathbb{R}^n$, and write $x_n = x \cdot e_n$ and $x' = x - x_n e_n$, where $e_n = (0, \ldots, 0, 1)$ is the nth unit vector in the standard basis for $\mathbb{R}^n$. Define an infinite open cone K for $a \in (0, 1/2)$ by $K = \{x \in \mathbb{R}^n\colon |x'| < a x_n\}$. In our problem (1.6), let Ω is a subset of K, $0 \in \partial\Omega$ and $e_n \in \Omega$. Construct a function $w : \overline{K} \to \mathbb{R}$ by $w(x) = (|x'|^2 - a^2 x_n^2)/[2(n-1)(n-2-2a^2)]$. Then $w(x) \leqslant 0$ for all $x \in \overline{K}$ and consequently $w(x) \leqslant 0 = u(x)$ for all $x \in \partial\Omega$, and direct calculation shows that for $x \in \Omega$, $S_2(D^2 w) = 1 = S_2(D^2 u)$. So the comparison principle implies that $u(x) \geqslant w(x)$ for all $x \in \overline{\Omega}$. For $t \in (0,1]$, let $x = t e_n$. Then $x \in \Omega$ and so $u(x) \geqslant w(x) = -a^2 t^2/[2(n-1)(n-2-2a^2)]$. Hence

$$\limsup_{t\to 0^+} t^{-1/\alpha} u(x) = 0$$

if $-\alpha^{-1} + 2 > 0$; that is, if $\alpha > 1/2$. Then Lemma 2 with $y = 0$ and $z = e_n$ shows that $-u$ is not α-concave for $\alpha > 1/2$, which means the index $\frac{1}{2}$ making $-(-u)^{1/2}$ strictly convex is sharp. □

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In this note we consider convex viscosity solutions to the quadratic Hessian inequality (1) Ïƒ2(D2u) â‰¥ 1. Our main result is their strict two-convexity. That is: Theorem 1.1. Let u be a convex viscosity solution to (1) in â„¦ âŠ‚ Rn, and let L be a supporting linear function to u in â„¦. Then. dim{u = L} â‰¤ n âˆ' 2. Theorem 1.1 is sharp in view of the example u = x21 + x22, with L = 0. Local smoothness of convex viscosity solutions to.Â In this section we recall a few classical facts about the k-Hessian equation. Below â„¦ denotes a bounded domain in Rn, and 1 â‰¤ k â‰¤ n. We rst recall some facts about the Ïƒk operator. The function Ïƒk on SymnÃ—n denotes the kth symmetric polynomial of the eigenvalues. Module â€" 2 Lecture Notes â€" 2. Convexity and Concavity of Functions of One and Two Variables. Introduction In the previous class we studied about stationary points and the definition of relative and global optimum. The necessary and sufficient conditions required for a relative optimum in functions of one variable and its extension to functions of two variables was also studied.Â To determine convexity or concavity of a function of multiple variables, the eigenvalues of its Hessian matrix are examined and the following rules apply. (a) If all eigenvalues of the Hessian are positive the function is strictly convex. (b) If all eigenvalues of the Hessian are negative the function is strictly concave. (c) If some eigenvalues are positive and some are negative, or if some are zero, the.Â Solution. Î"x f. Keywords: Nonlinear elliptic equations; Convexity of solutions 1. Introduction The convexity is an issue of interest for a long time in partial differential equations, it is intimately related to the study of geometric properties of solutions of general elliptic partial differential equations. It was Gabriel [8] first obtained that the level sets of the Green function in three-dimension convex domains in R3 are strictly convex. Makar-Limanov [18] considered the following elliptic boundary value problem: u = âˆ'1 in Î©, (1.1) u=0 on âˆ‚Î©, * Corresponding author. E-mail addresses: xinan@ustc.edu.cn