WHAT IS A PATTERN?: AN ELEMENTARY TEACHER'S EARLY EFFORTS TO TEACH MATHEMATICS FOR UNDERSTANDING

Ruth M. Heaton[1]

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INTRODUCTION

I taught elementary students mathematics the way I learned it. My mathematics instruction was rule driven, procedure based, and algorithm-oriented. In what would have been my tenth year of elementary teaching, I became a full-time doctoral student in teacher education at Michigan State University. Through work as a research assistant on a project designed to study the relationship between educational policy and teaching practice in the areas of mathematics and literacy,[2] I expanded my vision of mathematics education. Through work on this project, I became acquainted with state- and national-level documents calling for a reform of mathematics education (California State Department of Education [CSDE] 1985, 1992; National Council of Teachers of Mathematics [NCTM] 1989, 1991). As I observed elementary teachers and watched them try to make mathematics meaningful for students, I got my first glimpse of the difficulties experienced teachers face as they try to teach mathematics for understanding.[3] I contemplated what it might be like to teach mathematics, myself, in ways that probed and developed meanings of mathematical ideas. I wanted to try to understand, from the inside, what was so hard and challenging about the mathematics teaching these experienced teachers were aiming to do. What would it be like to try to change my own teaching practice?

Creating a Practice

In the following school year, the second year of my doctoral studies, 1989-1990, I arranged to teach fourth-grade mathematics, four days a week, one hour per day, in a Professional Development School. This was a school whose teaching staff were already working in partnership with faculty[4] from the College of Education at Michigan State University and who, for several years, had been exploring alternative ways of teaching mathematics. One or both of the parents of students in this school were students at Michigan State University and moved to Michigan from places all over the United States and the world

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[2]This project is known as the Educational Policy and Practice Study (EPPS), codirected by Deborah Ball, David Cohen, Penelope Peterson, and Suzanne Wilson.

[3]For numerous examples of the kinds of difficulties faced by experienced teachers I and other researchers on the EPPS project observed, see the following case studies: Ball 1990, Cohen 1990, Heaton 1992, Peterson 1990, Prawat 1992, Putnam 1992, Remillard 1992, Wiemers 1990, and Wilson 1990.

[4]Deborah Ball and Magdalene Lampert were among university faculty who had already worked for several years with teachers in this school. Both worked with teachers and students in the area of mathematics.

to attend the university. Of the 23 students in my class, 12 were English-speaking United States citizens. The other 11 were from various countries around the world and either entered the class speaking English or learned to speak it over the course of the school year. There were three adults who were also a part of my teaching. The classroom teacher, an experienced teacher responsible for teaching these fourth graders all other subjects, observed each day I taught, and we talked on a regular basis about what she noted (Heaton, Reineke, and Frese 1991). Jim Reineke, a peer and a doctoral student in educational psychology, observed and interviewed me from September to December for a study he designed around understanding a teacher's and her students' changing conceptions of mathematics. Magdalene Lampert, a university teacher educator and mathematics teacher, taught fifth-grade mathematics in the classroom next door and observed me teach twice each week. I observed in her mathematics class at least that often and regularly had conversations with her about teaching.[5]

I began the school year making use of Comprehensive School Mathematics Program (CSMP) (McREL 1986), a mathematics curriculum developed prior to the current wave of mathematics reforms, yet in many ways, the pedagogy implied within it seemed consistent with "new" ways of thinking about mathematics teaching. Problem solving is a central focus. Communication or mathematical discourse plays an important role. A major aim of the curriculum is to help all children learn to reason about mathematical ideas and understand how mathematical ideas relate to one another. CSMP is an innovative mathematics curriculum that looked different from any teacher's guide I had used in the past. Relying on it heavily at the start of the school year seemed like a safe and reasonable way to begin teaching mathematics in ways that seemed dramatically different from what I had experienced as a student or the kinds of mathematical experiences I had offered students as a teacher.

Studying a Practice

Three years after my efforts to teach differently, I considered options for my dissertation research. I returned to the audio, video, and written documentation I and others had collected around my mathematics teaching. At the time of my teaching, like the teachers I had earlier observed, I found the teaching I aimed to do challenging and difficult. Studying

[5]See Heaton and Lampert 1993 for further discussion and analysis of our relationship around my learning to teach.

my own efforts was a way to understand and to communicate to others interested in the enactment of these reforms what is so demanding about trying to teach mathematics in new ways. The value of this study lies in its contribution to understandings about teacher knowledge, learning, and development around these reforms in mathematics education from a perspective rarely found in educational research, from a learning teacher's point of view.

My study complements much of the existing subject-specific case research on teacher knowledge (i.e., Ball 1993a, 1993b; Grossman 1990; Gudmundsdottir 1990; Wilson and Wineberg 1988) grounded in the practices of experienced teachers. This research cutting across subject areas has led to new practical and theoretical (i.e., Shulman 1986) understandings of teacher knowledge and its use in teaching practice. There is a body of research on teaching concentrated in the area of mathematics. Cobb et al. (1992) and Yackel, Cobb, and Wood (1991), for example, have attempted to clarify what it means to teach mathematics for understanding from cognitive and sociological perspectives through an analysis of interactions. Schoenfeld (1988) focuses on the influence of social context as well as what it means to do mathematics (Schoenfeld, in press). Lampert's work (i.e., Lampert 1989, 1990) includes attention to choosing and posing mathematical problems, developing mathematical tools for facilitating communication between the teacher and students, and understanding mathematical knowledge in the context of classroom discourse. Ball (1993b) has looked closely at instructional representations, representational contexts, and pedagogical content knowledge in the context of mathematics teaching. Taken together, this research deepens our understandings of what it means to teach for understanding. My study enriches this research on teaching mathematics for understanding. What my work does is highlight some of the challenges of *learning* to teach mathematics for understanding or learning to bring together, in practice, mathematical understandings of content with the development of a new pedagogy. It sheds light on the problems, dilemmas, and difficulties associated with a teacher's efforts to understand and make use of the interdependence between content and pedagogy while guiding classroom interactions, choosing appropriate mathematical tasks, making mathematical tools and representations accessible, and being responsive to students.

The process of learning to teach remains somewhat less explored and theoretically less developed than current understandings of what it means to teach. There is some research in the area of learning to teach in the context of mathematics education that spans both the practice of preservice and beginning teachers (Borko and Livingston 1989; Eisenhart et al.

1993; Schram 1992) as well as veteran teachers attempting to make changes in practice (i.e., Cohen et al. 1990; Featherstone, Pfeiffer, and Smith 1993; Hart 1991; Schifter and Fosnot 1993; Simon and Schifter 1991; Stein, Grover, and Silver 1991; Thompson 1985; Wood, Cobb, and Yackel 1991). A general theme across all of these studies is the difficulty of learning to teach mathematics in meaningful ways whether one is a novice teacher trying to develop a practice or an experienced teacher trying to change one. What remains little understood is what makes learning to teach or changing one's practice so hard. My study offers insight into the difficulties from the perspective of a teacher who lived them.

In many ways, I represent a best-case scenario of teacher change or learning. First, while my ability to understand the mathematics in children's understandings as well as the curriculum appeared lacking at the time of my teaching, it is important to know that I have taken more mathematics coursework than most people who opt to be elementary teachers. I had successfully completed mathematics classes through calculus. Second, I have received public recognition for my work as a teacher. In other words, I am a good teacher. Any tendency to want to dismiss my difficulties because of my general weaknesses as a teacher is inappropriate here. Third, I tried to make change under extraordinary and nearly ideal conditions when compared to the ways in which the work lives of elementary teachers are usually organized and the kinds of support which are typically available.

BEGINNING EFFORTS TO CHANGE MY PRACTICE

What I present here is the first of four teaching events, spanning the entire school year, which I analyzed as a part of a larger study.[6] The different teaching events feature different lengths of time: a day, a moment, a series of days, a series of days followed by a six-week break, and then the second to last day of math class for the school year. The difficulties and confusions I confront in the teaching event described and analyzed here are based on a lesson (9/20/89) in the third week of the 1989-1990 school year. The challenges and frustrations that arise in this lesson are representative of what I had been facing on a daily basis since the school year began. The first few weeks of teaching were wrought with difficulties and frustration as I tried to enact the vision I held for the kind of mathematics teaching I wanted to be doing. I continually felt as if I was falling short of my goal. I began

[6]See my dissertation (Heaton 1994) for an analysis of my learning around all four teaching events.

the year heavily dependent on the teacher's guide, following it much the same way I have followed mathematics textbooks in the past. I thought making use of an innovative text would lead to innovative teaching. I was wrong. Making use of a new textbook in old ways was not the solution. Over time, I had to learn to construct a new relationship with the teacher's guide. I needed to reconsider what a guide could represent, what it is good for, what I expect of it, and how to make use of it. The description and analysis of this teaching event illustrates the start of that learning process.

I began with my plans for teaching this specific lesson. They include my view of the teacher's guide at the start of the school year, how I viewed an enactment of the reforms relative to this particular lesson, and an overview of the mathematical tasks I wanted students to do. I provide a detailed account of what happened as students worked the problems focused on patterns and functions which I posed. Drawing on my journal, Maggie's observation notes, and my conversations with Jim, I offer some possible explanations for the frustrations I felt during this particular lesson. I revisit these frustrations three years later, bringing fresh eyes to my teaching and new insights to what it was that I was learning. I conclude by drawing parallels between what I am learning from this particular math lesson about learning to teach and Mark Twain's adventures learning to navigate a riverboat as portrayed in *Life on the Mississippi* (Twain [1883] 1990).[7]

Following the Teacher's Guide

The lesson I did with students on this particular day came from "The World of Numbers," one of the content strands that make up the spiral curriculum of CSMP. At the time of this lesson, I had been following the spiral organization of the CSMP teacher's guide from one strand to another, from one lesson to the next, with little continuity across lessons. As I looked over this particular lesson and noticed the title, "Composition of Functions," I wanted some sense of what this lesson was intended to be about as well as what students were supposed to do.

[7] In trying to understand my own learning in the instance of practice featured here and in the other instances found in my dissertation, I looked for metaphors in texts about others' learning from fields quite far from educational research. These texts included Twain's ([1883] 1990) writing about learning to navigate a steamboat on the Mississippi River, Sudnow's (1978) writing about learning to play improvisational jazz, and Novack's (1990) study of improvisational dance. Their relevance to, yet distance from, mathematics teaching made them exciting and powerful analytic tools for me. The descriptions and analyses of learning experiences offered by these writers became useful in helping me to create new categories, images, and language for understanding and communicating my own experiences learning to teach.

There had always been a clear match between what the math textbook[8] said I was supposed to teach and what students were to learn and do. In my past teaching, what my students were supposed to learn was what I taught them to do. For example, if students were supposed to learn to divide, I taught them the rules and procedures for computing the algorithm, and they practiced finding the answers to many long-division problems. Knowing when and what they understood was also more straightforward. Answers were either right or wrong. If too many students had difficulty, I repeated my explanation of the rules and procedures. In my past teaching, the mere title of a math lesson (i.e., two-digit division, three-digit multiplication, addition of fractions) gave me sufficient information to plan, teach, and assess the successfulness of any lesson. There was never a question about what I was supposed to teach, what students were to learn, and whether or not we had accomplished what was planned.

In this lesson on the composition of functions from CSMP, the relationship between what to teach, what was to be learned, my role, and what students were supposed to do did not seem so clear to me. I saw a need to construct connections between the activity in the teacher's guide and the mathematical ideas the activity was intended to teach. The title of this lesson, the "Composition of Functions," gave me little information about either. The language was foreign to my past experiences as an elementary teacher. In my nine years of teaching, I could not recall any topic, chapter, or lesson in an elementary math textbook that used this same language.[9] While I had learned something about functions somewhere in my own learning of mathematics, what and why they were to be a course of study for fifth graders was a mystery to me.

I turned to the lesson summary, a description found at the start of each new lesson in

Using arrow diagrams and the Minicomputer, investigate the composition of certain numerical functions, for example, +10 followed by +2 and 3 x followed by 2 x. (P. 11).

[8]I use textbook and teacher's guide interchangeably through this text. The CSMP curriculum is a collection of teacher's guides. There are no student textbooks as are found in most more conventional mathematics curricula.

[9]Leinhardt, Zaslavsky, and Stein (1990) note the marginality of functions as topic in elementary math textbooks. Open Court's *Real Math* (Willoughby et al. 1981) and CSMP (McREL 1986) are among the exceptions.

the teacher's guide.

Reading this summary generated more questions for me than it answered. I knew what it meant to find an answer to a math program, but what did it mean to investigate a mathematical idea? What did it mean to investigate the composition of certain numerical functions? What was there to investigate? What were we supposed to be looking for? How was I supposed to assess what was learned in the investigation? Were there right and wrong ways to investigate? Finding the right answer to a problem had signaled being done in the past. How would I know when we were finished? How would I know when it was time to move on?

In an attempt to further understand what was meant by the phrase "composition of functions," I read through the content overview of "The World of Numbers" in the section entitled "Composition of Functions."

> Several lessons in this strand deal with what happens when you compose a sequence of functions, that is, apply the functions in order one at a time. These compositions lead to many general, powerful insights into the properties of numbers and operations. (P. xix)

After reading this, I questioned what sort of powerful insights we were searching for and why. What constituted a "powerful insight"? I read a bit further and found that these students, in previous years, should have had experiences with the composition of functions.

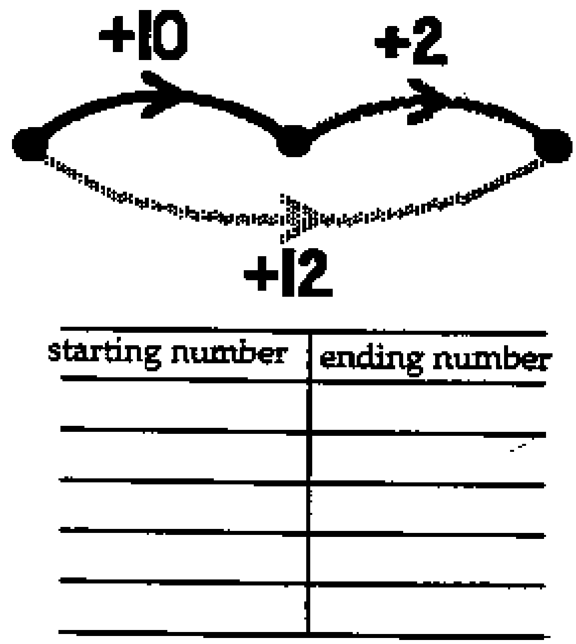
> Your students' extensive experiences with the composition of functions in the CSMP Upper Primary Grades curriculum led them to many insights that involved the development of algorithms, the discovery of number patterns, and efficient mental arithmetic techniques. A goal in this strand is to review these discoveries and to apply composition to new situations and problems. (P. xx)

The fact that students might be somewhat familiar with the content of the lesson I was about to teach was of some comfort. But, if this was true, would they remember what they had learned? And, what about the handful of students in my class who were new to the school? What could I count on in terms of any students' past experiences?

Most of the CSMP lessons were designed to be completed in a single class period. In the first few weeks of my teaching that year, I found that lessons took much longer than CSMP or I planned. CSMP typically allowed one day for each lesson while I spent at least two days, sometimes longer, on each one. I had begun to feel like I was falling behind. I looked for logical ways to divide the single day CSMP lessons over a couple of math classes. One way to lessen the feeling of falling behind was to become more realistic about moving ahead.

This particular lesson on functions involved a sequence of problems around addition, subtraction, and multiplication. I decided to do the three problems from the teacher's guide that dealt with addition and subtraction (defined as Exercise 1) one day and multiplication (defined as Exercise 2) the next. Even with splitting this lesson over two days, I remained skeptical as to whether or not I would actually get through what I had planned. These days, my judgements about pace felt off. This was unsettling. As I found myself falling further behind in terms of numbers of problems finished and pages covered in the teacher's guide, I began to question if these were reasonable ways to measure progress in my current practice. Why were things taking so much longer? What were realistic expectations? Was I moving anywhere? In those first few weeks, I experienced a strange sense of feeling behind in a context where I felt unsure of where I was supposed to be going.

The problems in the teacher's guide under the heading of "composition of functions" looked like the example in Figure 1. The labels of the arrows varied from one problem to the next. At the time I planned the lesson, the table seemed easy enough for students to complete. I was certain they could follow the arrows, one red and one blue in the teacher's guide, and come up with numbers to fill in the table. Once the table was filled with numbers, the students were to notice patterns in the relationship between the input (starting) numbers and the output (ending) numbers. I decided to do this first problem together with the whole group and assign the remaining two problems in the teacher's guide for the students to work on their own.



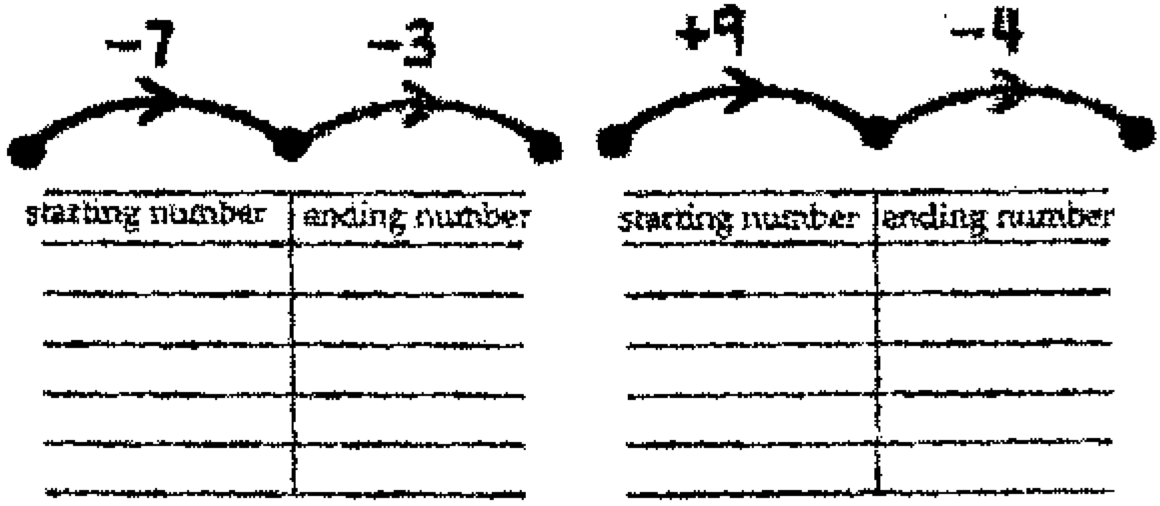


Figure 2

I planned to do just what appeared in the teacher's guide. Students would find starting and ending numbers to complete the tables, identify the composition arrow, an arrow which would encompass the other two, for each table of numbers, and notice patterns.

Plans to Enact a Vision of the Math Reforms

In my early weeks of teaching, CSMP was a guide for my practice as were ideas that I got elsewhere for the kind of math teaching that I wanted to learn to do. When I came across the question of patterns in the script of the CSMP lesson, my familiarity with the math reforms at the time, based on my observations in Maggie Lampert's fifth-grade math class and what I had read in the *Mathematics Framework* (CSDE 1985), helped me to see this as an important question even though a study of patterns was foreign to my past experience as a teacher or student. The importance of the study of the relationship between patterns and functions was described this way:

> The study of mathematical patterns and functions enables students to organize and understand most observations of the world around them. It involves discovery of patterns and relations, identification and use of functions, and representation of relations and functions in graphs, mathematical sentences or formulas, diagrams, and tables. (P. 10)

I had observed Maggie engage fifth graders on a regular basis in discussions of patterns. I was eager to have the opportunity to attempt a discussion of patterns with these fourth graders. I had become intrigued with the idea of mathematical discussions and wanted to get students talking. I thought patterns would give us something to talk about.

Before I became acquainted with the reforms, I never knew there was anything that would lend itself to a discussion in math class. I had learned through observing Maggie's class and reading the *Mathematics Framework* that it was possible to discuss mathematical ideas. I could ask students to explain how they solved a problem and why they solved a problem in a particular way. I had led discussions in other subjects[10] and could not imagine that mathematical discussions would be that much different to conduct, now that I knew there were ideas to discuss.

[10] I engaged students in discussions of real literature in language arts and held discussions in social studies making use of the *Man: A Course of Study* (MACOS) curriculum materials.

At the onset of this school year, I expected that moving away from a primary focus on rules and procedures would deepen students' understandings of mathematics and push at the limits of my own mathematical experiences and understandings. I was worried about the mathematics I did not understand. I was reminded of the limits of my understanding as I searched for the meaning of "composition of functions" within the text of CSMP. These concerns about my lack of mathematical understanding did not, however, keep me from going ahead with this lesson. With the teacher's guide in hand, I felt more or less prepared. Even though I was unclear exactly what I was supposed to teach, I thought having something to do would get us through the hour. The teacher's guide provided me with problems I thought students could do, questions for me to ask, and even student responses I might hear. I hoped that working on three different tables would fill the time and asking the question of patterns would encourage a different way of thinking about numbers and spark a discussion—two aspects of the reforms I was aiming to make prominent in my practice that day.

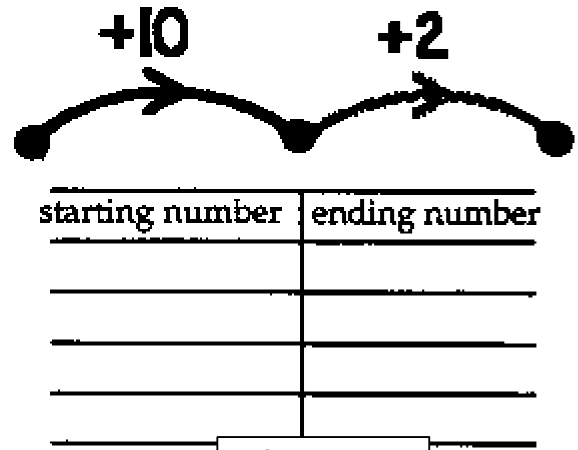


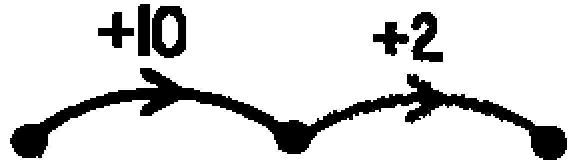
Figure 3

A Table of Numbers

At the start of class, I wrote the problem in Figure 3 on the chalkboard. I said, "I'd like you to take a look at the chart. Think of a number. Everyone just think of a number." Bob[11] inquired about the parameters for choosing a starting number. "Between what?" he asked. We barely began and already there was a question. I had not considered putting conditions on the problem nor was there any indication in the teacher's guide that I should. Why was Bob asking the question? Given my own uncertainty about why we were doing what we were doing, the last thing I wanted was for a student to question me about the task. I said to him, "Just think of a number and add 10 to it and then add 2 more to it." After a few moments of silence while students thought of numbers, I asked for someone to give the beginning number they chose. I started with Richard, whose hand was in the air.

[11] All students are identified with pseudonyms.

Richard: 99
Ms. Heaton:[12] And what did you end with?
Richard: 111
Ms. Heaton: And can you tell us how you got that?
Richard: Well, I did 99 plus 10 equals 109 and then plus 2 is like 9 plus 2 equals 11We followed a similar routine for adding the first four pairs of numbers to the table (Figure 4). Asking students the question, “How did you get that?” was my idea. It did not appear in CSMP, but it was a question I had heard Maggie ask in order to prompt her fifth graders to talk. Rather than just have students give me beginning and ending numbers, I added this question thinking that asking it would get students talking about how they solved the problem. What happened was that individuals, like Richard, responded with procedural explanations that did not seem to hold anyone’s interest. Why was this happening? When Maggie asked the question of her students, they always offered interesting explanations.



starting number	ending number
99	111
8000	8012
250	262
1	13

Figure 4

As the teacher’s guide suggested, I varied the process of filling the table with numbers by asking some students to give ending numbers first, followed by an explanation of how they generated them. The teacher’s guide also suggested that I call on someone different to offer these explanations. These variations in the task and ways of calling on students appealed to me. I thought that starting with the ending number might make the explanations about how students got from one number to the other more interesting and calling on a variety of students would increase the number of people talking. Maybe this would help the talk seem more like a discussion in which many were engaged. Here is a sample of the interactions that occurred.

Ms. Heaton: Can somebody give us their ending number? The number they ended with? David?

[12]Please note that I refer to myself as “Ms. Heaton” in the excerpts of transcripts. I do this because this is how I was referred to by students. In excerpts of transcripts of conversations with Jim Reineke, I refer to myself as “Ruth.” Jim and I are peers and refer to one another by first names. This switch in how I refer to myself is a complicated issue and related to issues of voice and identity. For further discussion of these issues, see Chapter 3, “Learning About Learning to Teach” (Heaton 1994).

David:	19
Ms. Heaton:	And what was your starting number?
David:	7
Ms. Heaton:	Can you tell us how you got that?
David:	Because 7 plus 10 is 17 and 17 plus 2 is 19.
Ms. Heaton:	Okay, can someone else give an ending number?
Jennifer:	215
Ms. Heaton:	Can someone else tell us what she might have started with? Bob?
Bob:	203
Ms. Heaton:	Okay, how did you get that?
Bob:	Because 203 plus 10 is 213, 213 plus 2 is 215 and I have another number.

I continued calling on students and filled in starting and ending numbers until I ran out of space on the chalkboard. We ended up with a table of numbers (Figure 5), but my efforts to generate a discussion led us nowhere.

starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4000~~ 4988	5000

Figure 5

Do You See Any Patterns?

Disappointed with the lack of interesting talk that had occurred thus far, I held high hopes for the next part of the activity in which I planned to ask students to notice patterns. I followed the script in the teacher's guide and said to the students, "I want you to look at these numbers. Do you see any patterns?" For the next 15 minutes, students proceeded to give their ideas about patterns. I called on Valerie first. She said, "Each of them have a beginning number and then they have an ending number that is 12 more." I asked her how she knew the ending number was 12 more. She said, "Because you have to add 10; you find a number, you add 10 to it and then 2." I added the green arrow, the composition arrow, and said, "If I were to put in another arrow here (Figure 6), what would I put, plus what?" Valerie responded, "Plus 12." I added the label to the arrow and, given my goal to initiate a discussion, proceeded to find out what others were thinking.

Pili said, "I see, I think I see a pattern. It is 80, 80." She pointed to the 80 in 8000 and the 80 in 8012 as she said, "Eighty right here and 80 right here." Hearing Pili's idea, I wondered what sort of patterns we were looking for. Definitely not ones like this. She was right. There was an 8 and a zero in each of those numbers but taken together, they did not mean "80." The 8 was actually 8000 and the zero was in the hundreds place. In another attempt to promote discussion, I asked students to comment on Pili's idea. This call for comments from other students was a pedagogical move I had observed Maggie make with her students. In addition to being able to explain their own ideas, I wanted to teach students to listen and build on the ideas of others. I thought that what Pili had said seemed like a meaningless pattern and I hoped that one of her classmates might say something to challenge her idea.

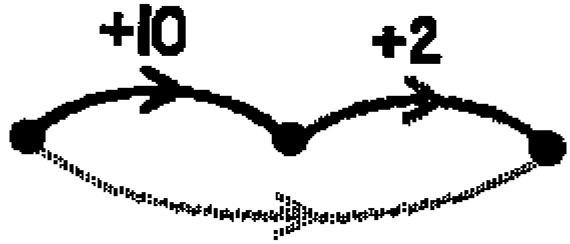
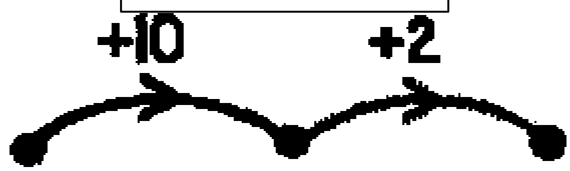


Figure 6

I repeated words not found in CSMP but ones I had often heard Maggie ask. "Okay, what do other people think about that?" I called on Jennifer, who said, "I agree with her." This, unfortunately, was not the response I wanted. I hoped Jennifer would disagree with Pili. I could have asked Jennifer why she agreed with Pili, but I was not in a mood to hear explanations for what I thought was a meaningless pattern. Wasn't it clear that Pili was

Figure 7



starting number	ending number
99	111
[redacted]	
250	262
1	13
7	19
203	215
~~4000~~ 4988	5000

wrong? Should I just come out and tell her so? But wasn't doing that counter to my goal for students to construct their own meanings? How did the categories of "right" and "wrong" fit into my desire to give students space to make their own sense of ideas? Should I accept all of students' ideas, even the ones I thought were wrong?

I moved on without pursuing Jennifer's comment or dealing with this bigger issue of right and wrong answers. I addressed the class, "Okay, are there any other patterns that you see?" I called on Lucy, who said, "There is 8 and then 8 going across and then 2 and then 2 going across." She went to the chalkboard and pointed out what she was talking about. She pointed to the 8s at the beginning of 8000 and 8012 and the 2s at the beginning of the pairs of numbers 250, 262 and 203, 215 (Figure 8). She was right. These numerals matched one another, but what did this observation have to do with the question of patterns in the context of functions? Reluctantly, I returned to Pili who was waving her hand frantically, ready to give another "pattern." She said, "I see three zeroes here and here," and she underlined the three zeroes in 8000 and 5000 (Figure 9). I was worried. This "pattern" seemed even more irrelevant than her last one. Maybe I should have come out and said what I thought of her ideas. She attempted to draw connections between a pair of numbers not even in the same row. Since the ordered pairs were generated independent of one another, I knew there was no mathematical reason why there would be meaningful connections between numbers not

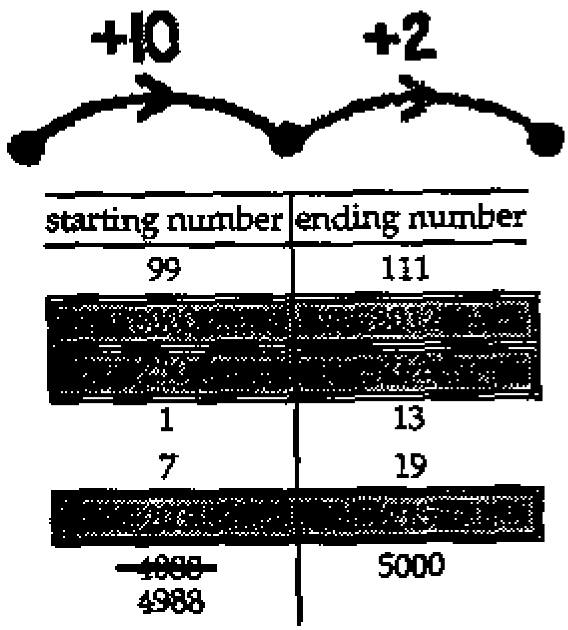


Figure8

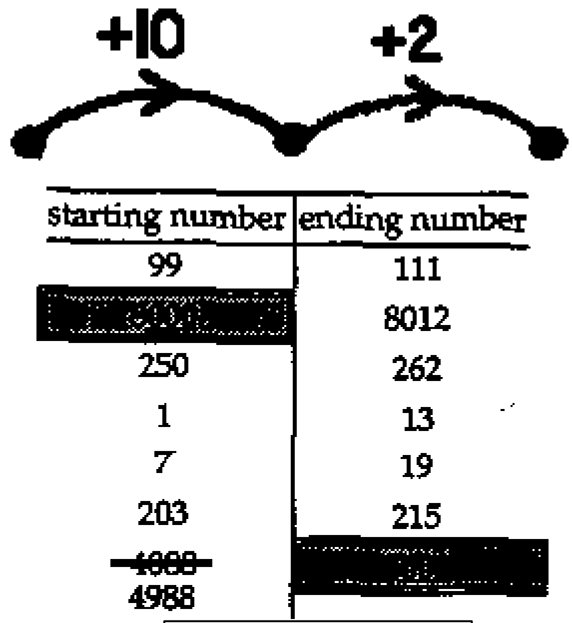
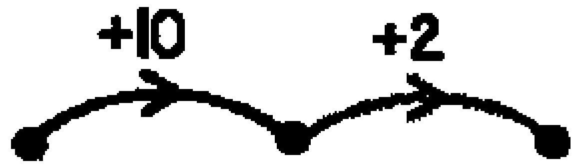


Figure 9

found in the same row. Maybe Pili would see for herself the unreasonableness of what she was doing if I could get her to talk about her idea. I asked her to explain. She said, “In 8000 there are three zeroes and in 5000 there are three zeroes.” But what did this have to do with patterns? What was a pattern? I knew that what I was hearing were not patterns. Jennifer followed Pili’s comment and said, “I have something similar to hers, but it is not exactly the same.” She went to the board and underlined the zeroes in the hundreds’ place in 8000, 8012, 4088, and 5000 (Figure 10).

I felt a mix of emotions. On the one hand, I was excited that a student had listened to another student and was making an attempt to build on a classmate’s idea; on the other hand, I was growing frustrated as I watched one irrelevant pattern lead to another. In fact, Jennifer just included a number that did not even belong on the chart. It was there with a line through it because it had been revised. The idea of taking all ideas seriously, revised or not, was something else I had picked up from Maggie. Seeing Jennifer making use of the error, however, made me question my decision not to just erase it like I would have done in the past. Was that revised number just adding to the confusion and the fact that it was revised insignificant to what we were doing? Were there some times when visible evidence of revised thinking was more important than others?



starting number	ending number
99	111
[redacted]	
250	262
1	13
7	19
203	215
[redacted]	
4988	

Figure 10

These questions about revisions seemed like the least of my worries. How could I stop this generation of meaningless patterns without just reverting back to my past ways of telling students they were wrong? I was trying to search for mathematical sense in what students were saying, but was this an instance where there was none to be found? I was curious how Jennifer would describe the pattern she noticed, so I asked, “What would you say that pattern is? How would you describe it?”

Jennifer:	One here and one here, and one here and one here.
Ms. Heaton:	I don’t understand what you mean by one here. What are you showing me?
Jennifer:	There is a zero in the hundreds’ place and one zero in the hundreds’ place, and there is one zero in hundreds’ place and one zero in the hundreds’ place.

Hearing Jennifer talk of place value gave me a momentary sense of relief from the meaninglessness of all of this. I continued to ask myself, however, whether any of what she or the other students were saying had to do with patterns or functions. What was a meaningful pattern? What was a function? I called on Richard. He explained, “I have another pattern. Can I go up there? Right here is three zeroes in a row and three ones in a row, and then three zeroes in a row.” I asked what the pattern was. He said, “000, 111, 000” (Figure 11). I felt troubled as I listened to Richard. I knew these were not patterns in a mathematical sense but what were they? And, more importantly, what were mathematically meaningful patterns in this table of numbers? These students noticed “patterns” in pairs of numbers in the same row, pairs of numbers that spanned rows, and now senseless patterns in numbers in each of the columns. I knew that “meaningful” patterns could not be found in columns of numbers in which pairs of numbers had been generated by beginning or ending numbers chosen at random.

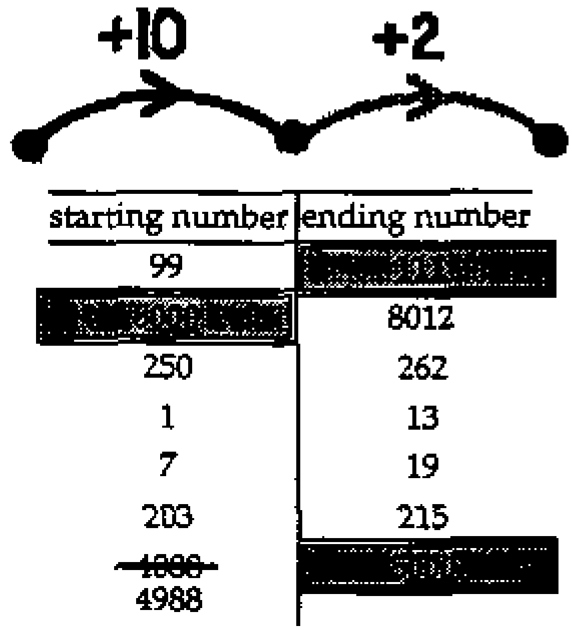


Figure 11

After more than 15 minutes of these patterns, when I could bear this “discussion” no longer, I announced, “I want to move on to another example.” I wrote the following problems (Figure 12) on the chalkboard for students to work on.

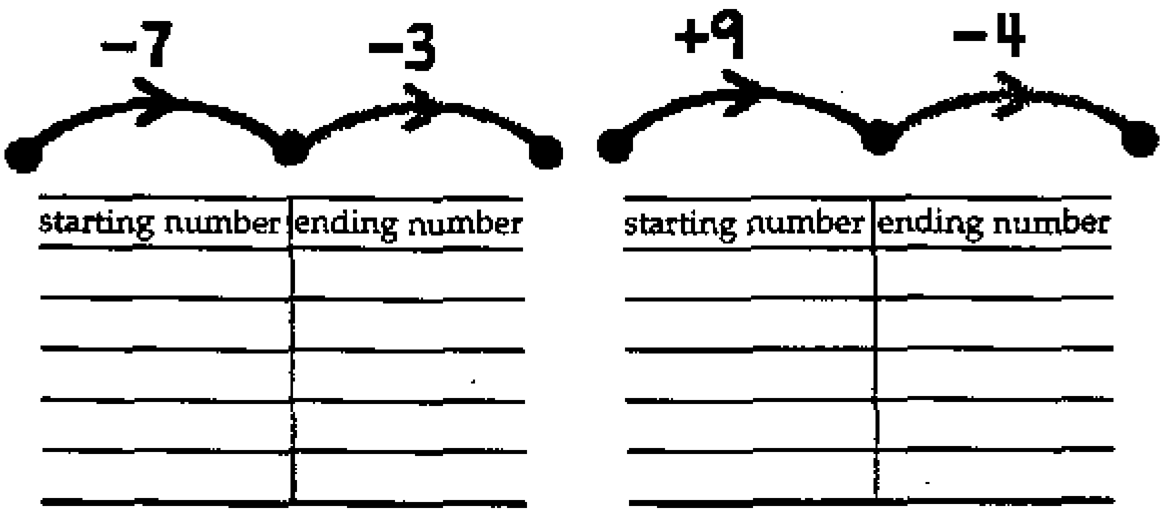


Figure 12

As the students worked on these problems, I had a few moments to think and asked myself once more, What is a pattern? I knew that I had already made plans to teach the second half of this lesson the next day. Could I bring myself to come back the next day and face this teaching and, quite possibly, the question of patterns again? I left school feeling perturbed with myself for having trusted the teacher's guide and annoyed with myself for having thought I entered this lesson prepared to teach it.

REFLECTING ON TEACHING

That evening, I was distraught and troubled by the day's events. My mind was filled with questions as I tried to sort out the difficulties I experienced that day. Was I teaching? What were students learning? What were they supposed to be learning? What would teaching and learning in this situation look like? Why was I unable to get an interesting discussion started? Why was I asking students the question of patterns anyway? What was a pattern? Did the patterns students noticed have any relevance? What is a function? Why didn't students come up with responses that matched the teacher's guide? How was I supposed to be making use of this teacher's guide?

In my interview with Jim Reineke that afternoon, I said:

> I felt like I was floundering today . . . I was asking myself the question, What do I mean by a pattern? *What does the word "pattern" mean? I am not sure I have an answer to these questions, but that's what I was thinking as I stood up there, listening to these kids.* And the sort of student responses the book says will come up—that CSMP puts in, what kids would respond when you

> say, "What's a pattern?"—none of that came up. I felt like I didn't know what to do to move a discussion about patterns. (Reineke's interview of author, 9/20/89)

I had trusted that the teacher's guide was going to help me with this work. I felt like it was letting me down. It was, at once, too much of a guide and not enough of a guide. It had given me enough guidance to lead me to believe we could do this activity without providing some broader sense of its purpose. The only reason I had for doing the lesson was because it was the next thing in the teacher's guide. In the midst of teaching, I found myself feeling lost and helpless with no way to respond to students.

As soon as I heard students' responses, I realized I did not know what I meant by "pattern" or really why I was asking the question. I initially asked the question because it was in the teacher's guide and, given what I knew about the reforms, it had seemed like a good idea to me. *But students' responses did not match what the teacher's guide had predicted.* It frustrated me to have an intuitive sense about what was not a pattern but no real sense of what would be sensible patterns or how to get students to notice and discuss them.

The predicted responses in the teacher's guide were not useful to me in the situation. Here is what appeared in the script of the teacher's guide.

> T: Look closely at this chart. What patterns do you notice?

> S: An ending number is always larger than the starting number.
> S: If you start with an even number, you end with an even number. If you start with an odd number you end with an odd number.
> S: An ending number is always 12 larger than the starting number (p. 12).

I had these responses in hand when I went into class. In the past, I would have been intent on listening for these responses. If I had not heard them, I would have encouraged students to produce these "right" answers. Like many good "traditional" teachers, I have had years of experience asking convergent questions. I know how to lead students to the "right" answer. But, in this situation, I wanted to be open to the idea that students' answers might vary from the teacher's guide. How was I to know which students' responses were reasonable variations of the responses found in the script and which were not? How would I figure out how to be responsive to these variations? Was there a way that I could be respectful of all students' ideas and at the same time let them know that many of their observations were irrelevant to the question of patterns? How could I respond to students' ideas in ways that would push their thinking? The script in the CSMP teacher's guide was designed with questions for me to initiate, but I felt frustrated by what little help it offered me in figuring out what to do next in the situation, especially when it seemed that what students were saying did not match what the developers of CSMP had anticipated they might.

My frustrations went beyond students' responses not matching the ones in the teacher's guide. I was bothered that the talk that was happening felt out of my control. Moreover, I did not have any sense of what it would mean for it to be *in* my control. I wanted to do something to guide a discussion but, not knowing what to do, I remained silent. Inside, I was struggling. I wanted us to have a meaningful discussion. I knew that we were not having one, yet I did not know what we could have talked about or how I could have shaped what my students were saying into a meaningful discussion. While troubled by my students' responses to the question of patterns, I was also wrestling with my own response. I asked myself over and over again, What is a pattern? I had not seen this question asked or answered in the teacher's guide, but it was a question that concerned me.

The script in CSMP offered me no assistance with responding to the students or to the questions I asked myself. If I had been more certain of the point of the lesson, I might have been more willing and able to move away from the script and ask the students a different question—*my* question—or I might have had a way of responding to their responses. As it was, the script in my hands, intended to help me in my role as the teacher, felt disconnected from my students and myself and the sense any of us were making of the question of patterns.

Questioning the Question

When I asked students if they noticed any patterns, my response was to ask myself, What is a pattern?[13] I asked students one question—What patterns do you notice?—and myself another—What is a pattern?—repeatedly throughout this lesson, yet I never

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[13] Recognizing the question of "What is a pattern?" in the context of asking the question of noticing patterns as well as struggling with the fundamental nature of that question seems similar to the question of "What is a whole number?" which comes up in a lesson one week later. Maggie and I wrote about that lesson (9/27/89) in "Learning to Hear Voices: Inventing a New Pedagogy of Teacher Education" (Cohen, McLaughlin, and Talbert 1993).

considered asking students the question I asked myself. I was surprised when Jim brought up the idea in our interview.

Ruth:	The question for me became, What is a pattern? And, Are these things that the kids are giving me patterns?
Jim:	So, why didn't you ask that?
Ruth:	Why didn't I ask . . . ?
Jim:	The class.
Ruth:	What?
Jim:	What is a pattern?
Ruth:	That would have been a good question. That is the question that I had on my mind.

As I said the last sentence, I heard myself laugh on the audio tape. *I think I was realizing how slow I was to see that what Jim was suggesting was that I ask students the question that had been on my mind.* It did not occur to me, prior to this moment in my conversation with Jim, that asking it would have been a reasonable thing to do. I did not even consider it an option. Why? One reason might have had to do with my concern for time and my own need to get through the lesson I had planned. How could I have justified spending time on what patterns were when, according to CSMP, students were supposed to be searching for them? The teacher's guide defined the task as noticing patterns, not defining them. Would the students have gotten through the task of noticing patterns if we had stopped to define them? I began, however, to question how worthwhile the task of noticing patterns had been without some shared understanding of what we were looking for and why.

I was relieved to find out that I was not the only one who questioned the meaning of patterns. Jim said that he had asked himself the same question. Maggie had also observed this lesson and, as I read her observation notes for the day, I saw that she had also been playing around with the definition of a pattern. What follows is an excerpt from her notes:

> Asking "Do you see any patterns?" What do you want to get out of that? Was the idea to get someone to say that the ending number is 12 more than the beginning number? There are relevant and irrelevant patterns (i.e., like 8-2-2 and 8-2-2). Hard to exactly explain what I mean by "irrelevant patterns." There *are* things you could get out of almost every pattern the kids come up with . . . What the kids are saying are more like *observations* than *patterns*. I think you were trying to get at this a bit when you asked, "Does that pattern apply to any other set of numbers?" (Lampert's observation notes, 9/20/89)

During the lesson, I assumed that I was the only one puzzling about patterns. It was comforting to know that I was not alone and that, from Maggie's perspective, there was not an easy answer. *The definition of a pattern is related to questions of relevance. Questions of relevance are related to questions of purpose and questions of purpose are related to the mathematics* (i.e., *both* mathematical ideas and ways of knowing) *to be learned.* Where might this discussion have gone if I had taken my own questions of patterns seriously? How might things have turned out if I had asked the question that made sense to me rather than the one with little meaning for me found in the teacher's guide?

I went into this teaching thinking that I did not know enough mathematics and that I was going to be learning a lot. That was in the abstract, before I started this teaching. Reality hit when I was faced with a question for which I really did not know the answer. Could I ask students a question that was genuine for me?[14] Admitting to myself and others that I had something to learn clashed with my view of my role and responsibilities as a mathematics teacher. I was the teacher. I was supposed to ask questions for which I had the answers. Or was I? *Could I legitimately ask a question as a learner while also being the teacher? Could I take the risk?* I needed to learn to set aside the teacher's guide, value what I did not know, and trust myself to ask a different question—even if it meant raising questions about questions in the teacher's guide.

The Limitations of Following a Teacher's Guide

My frustrations with the teacher's guide in this lesson prompted me to reconsider why it was I thought following this teacher's guide was a good idea in the first place. The content and pedagogy represented by the CSMP teacher's guide looked different from any curriculum materials I had used in the past. Since I wanted to move away from "traditional" teaching, "nontraditional" curriculum materials seemed like a useful tool for making fundamental changes in my view of content as well as how to teach it. I was coming to see that making the

[14] I had had the experience of asking genuine questions as a teacher and learner with students in the context of *Man: A Course of Study* (MACOS). For example, "What makes man human?"—the question around which the entire curriculum is organized—is a question for which I did not have an answer when I began teaching MACOS and one I continue to ponder. *Schoolhouse Politics* (Dow 1991), especially Chapter 4 on teacher workshops, supports the idea that other teachers found themselves, often accompanied by much anxiety, in a similar position.

changes in my teaching to achieve what I was aiming for was not going to be as easy as merely following the CSMP teacher's guide as I had followed teacher's guides in the past.

How was I making use of the CSMP teacher's guide in my teaching? As in the past, I used the CSMP teacher's guide to make my decisions about what and how to teach. In this lesson, what to teach was defined as the "composition of functions" and how to teach was to "investigate." I lacked a good understanding of both of these at the time. In my past teaching, what I did was the same as what I wanted students to learn and there was less room for interpretation of either. Now everything seemed wide open to interpretation and much more complicated.

With little success, I had tried to figure out the point of the lesson by using the teacher's guide when I planned this lesson. At the time, I ended up putting aside my questions about the mathematical idea of composition of functions and what students were supposed to learn and proceeded with giving students the problems from the text. The problems looked doable without an understanding of some larger purpose, and I trusted that doing them would lead to something meaningful. Once I asked the questions, I questioned my decision to continue discussing patterns without understanding why they were important: How could the teacher's guide have helped me to be better prepared for this lesson? Were there ways that these problems could have been situated in terms of broader mathematical goals that I understood? What would I have needed to know outside of the teacher's guide to have felt prepared and gotten the most out of these problems and the question of patterns? Were there aspects of this teaching that CSMP or any textbook could not prepare me for?

Another set of frustrations accompanied my use of the teacher's guide when I tried to generate a discussion and interact with students. The script of the teacher's guide offered me one possible route through the material. It offered enough detail of the activity to get students and I started even though I did not understand exactly where I was going or why. This had worked fine in my past math teaching, and it would have been fine this time if what had happened in my class had matched exactly what was anticipated by CSMP. But this was not the case. At the time, I thought there was a complete mismatch between the anticipated student responses in the teacher's guide and what students said. In my journal, I wrote:

> What my kids came up with did not resemble these responses . . . I am wondering what the role of the teacher's manual is in this type of teaching. A lot of the space in CSMP is taken up with examples of student/teacher

> interaction. I'm not finding this particularly useful and at times troublesome. When I read it, I think I have an idea of what is going to happen in class and today was an example, last Thursday was too, of things not going as I anticipated. I base my plans on the book but that doesn't seem to be working (Author's journal entry, 9/20/89).

This frustrated me to no end. Were there alternative routes to the one in the teacher's guide? If so, what were they, and how could I connect them to the varied sense students were making of the question of patterns?

Before this year, I would have steered us right back on the course defined by the textbook regardless of how students responded to my questions. Now, however, I wanted to be doing things differently. I wanted to be responsive to students' ideas. Was there a way I could do that and still get to where we were headed by choosing alternate routes prompted by students' sense-making? What sort of guide could give me enough of a feel for where I was headed to guide my decisions about how to get there, yet do so in ways that were responsive to these students' ideas?

What Is There to Talk About?

While I was teaching, I was frustrated with my attempts to generate a discussion. Even though students were talking, both as they filled the table with numbers and as they searched for patterns, it felt to me like there was nothing to talk about or what they proposed was meaningless. In the first part of the lesson, the discussion fell flat, and in the second part of the lesson, the substance being discussed seemed irrelevant. Teaching as telling dominated my past practice as a teacher. At the time of this lesson, only a few weeks into the school year, I was determined to change my ways of interacting with students and to avoid doing anything that resembled telling. I thought asking questions was a way to make this happen. I even inserted questions of my own into the script of the teacher's guide. There was still no discussion.

Given my lack of success at generating a discussion through questions, I began to think that there had to be more to my role than just asking questions. There was something to be learned about how to comment on students' ideas in the situation. How do I decide what to say, about what, to whom, and when? I discovered that I could get students talking with the questions I read about and heard Maggie use in her class. These questions—Do you see any patterns? How did you get it? What do others think?—along with the questions in the CSMP guide were pedagogical tools for getting people to talk. Asking them was a way for me to try on the role of a different sort of teacher. The questions got students talking but then what? *Asking the questions was one thing. Knowing what to do with the responses was quite another.*

I began to think that the decisions I could see I needed to make were related to where I wanted to go with a lesson which was dependent on an understanding of the purpose for attempting a discussion in the first place. If I had known why we were looking for patterns, I might have been able to consider the relevance of the students' patterns. When I asked a question about patterns in my journal, Maggie commented in the margin, "Isn't a larger question, Why are we looking for patterns in math class in the first place???" (Lampert's annotation of an entry in author's journal, 9/20/89).

A sense of purpose might have helped me to figure out what there was to talk about or possible ways to help students think about their ideas. I opened the discourse with questions I had heard Maggie ask, but I did so without a sense of where I wanted to go. I found myself with lots of students' ideas before me, plenty of my own thoughts in my head, but no way to respond, and the CSMP teacher's guide that carried me this far was of little use. I did not know what to do to make a discussion out of what students were saying. I did not know how to respond in ways that would push their thinking. I also lacked a sense of where I wanted to push their thinking and why. What was it that I was trying to teach about the composition of functions? What was important for students to learn? I did not have answers to these questions. I was trying to get a discussion of patterns going without a sense of what there was to discuss.

THREE YEARS LATER: REVISITING MY FRUSTRATIONS

At least part of my frustration at the time was directed at the teacher's guide and what I thought was its inability to help me respond to students. In going back to this lesson three years later, I wanted to understand more about my relationship with the textbook—what I expected of it or what I thought it would provide me with or do for me while I was teaching, the trust I had in it and the people who wrote it, and the way I made use of it in class. To do so, I have reexamined my frustrations and tried to understand what it is that I could have known or understood that would have eased or avoided my frustrations. What mathematics was I supposed to be teaching? What was mathematically significant about the patterns students gave me? What is a pattern? What is a function? What is the connection between patterns and functions? What is mathematically significant about the predicted student responses in the teacher’s guide? What was the intent of this lesson from CSMP’s perspective? How do the mathematical ideas in this lesson connect to broader mathematical ideas? How does doing the activity address what it is that students are supposed to learn? What is important for students to learn?

In the section that follows, I draw on resources outside of the CSMP teacher’s guide to answer these questions. It is unclear to me whether or not the authors of CSMP expected me to know prior to teaching this lesson what I now understand. From my conversations with people who know about CSMP and the written commentary on the curriculum, I have no doubt that the people who designed the individual lessons, the content strands, and the curriculum had a sense for the mathematical connections and relationships at many different levels that I am only now coming to understand. This mathematical structure, however, was not presented in ways that were accessible to me, nor am I certain that it was intended to be. It seems that the developers may have overestimated the sheer difficulty of the mathematical ideas represented by the curriculum.

In the course of pursuing an understanding of the mathematics for myself, I have discovered a set of connections and relationships among mathematical ideas in CSMP that were invisible to me at the time I taught the lesson. If the authors of CSMP thought I understood what follows, I am uncertain how they thought I would have learned it. If they assumed the information was in CSMP, it was not in a form that I found accessible. The areas of mathematics I pursue are organized under three broad questions. Why are patterns and functions an important topic to teach? Why teach about patterns and functions in a particular way? What sense might students make out of a study of patterns and functions?

Learning the Importance of Patterns and Functions

The presence of a lesson on the “composition of functions” within CSMP implies that it was something worth teaching. But why is this topic worth teaching? What is the importance of learning about functions? What is a pattern? Why study patterns? What is meant by the composition of functions? How do patterns and functions connect to broader mathematical ideas? Why does CSMP link patterns and functions? What is the relationship between patterns and functions? I had begun to ask myself various forms of these questions at the time of the lesson. Now, more than three years later, I have begun to construct some answers.

Why Spend Time on Patterns? I have recently begun to appreciate that mathematics is a science of patterns and a search for patterns drives the work of mathematicians. Steen (1990, p. 1) writes, “Seeing and revealing hidden patterns are what mathematicians do best.” In an effort to broaden and deepen students’ understandings of mathematics, reformers advise a similar search for patterns in mathematics by all students at every grade level (National Research Council 1989; NCTM 1989; CSDE 1985, 1992). For example, the *Professional Standards for Teaching Mathematics* (p. 4) notes that as “teachers shift toward the vision of teaching presented by these standards, one would expect to see teachers asking, and stimulating students to ask, questions like ‘Do you see a pattern?’” Patterns, according to the *Mathematics Framework* (CSDE 1992, p. 108), “help children to see order and make sense of underlying structures of things, situations, and experiences. Patterns help children predict what will happen.” A hunt for patterns expands the concept of doing mathematics beyond the search for a single right answer toward an understanding of mathematical relationships, an aim of the current reforms in mathematics education. Recognizing, describing, and creating a wide variety of patterns provides a foundation for exploring mathematical relationships in numbers in later grades. The *Curriculum and Evaluation Standards for School Mathematics* put it this way:

> From the earliest grades, the curriculum should give students opportunities to focus on regularities of events, shapes, designs, and sets of numbers. Children should begin to see that regularity is the essence of mathematics. The idea of a functional relationship can be intuitively developed through observations of regularity and work with generalizable patterns. (P. 60)

In the context of a study of patterns, the doing of school math bears strong resemblance to the work of real mathematicians. The study of patterns now permeates the latest version of the *Mathematics Framework* (CSDE 1992) as a “unifying idea” across content strands. There are good reasons why patterns have come to take a prominent role in elementary mathematics curricula.

> What matters in the study of mathematics is not so much which particular strands one explores, but the presence in these strands of significant examples of sufficient variety and depth to reveal patterns. By encouraging students to explore patterns that have proven their power and significance, we offer them broad shoulders from which they will see farther than we can. (Steen 1990, p. 8)

What Is a Pattern? There was no indication in the teacher's guide that my question about what a pattern is was reasonable. To have asked it would have required me to value my own question above what I found in the teacher's guide. This would have required a confidence that I did not have at the time. In the throes of changing my practice, my self-confidence was at a low point. I thought looking to the CSMP teacher's guide as an authority for what and how to teach was a safe way to proceed with making changes in my math teaching. Doing the teaching was hard enough without taking on all of the responsibility for decisions about what to do next. I assumed that the teacher's guide knew better than I what questions were reasonable to ask. At the time, I also assumed that my questioning the definition of a pattern was related to my being new to investigations of patterns. I realize now that noticing patterns and figuring out what a pattern is are related tasks. Contemplating one informs the other. Constructing the definition of a pattern, as you notice patterns, is part of the work of figuring out whether a pattern is relevant or irrelevant.

Regularity and predictability are the two fundamental characteristics of a pattern. Something is a pattern if you can observe regularity in it. The regularity allows you to be predictive about the pattern's behavior. A numerical pattern with regularity and predictability enables you to describe a relationship between two variables. *Identifying a pattern allows you to manipulate one variable and predict what will happen with the other. A relationship between two variables with this kind of regularity and predictability is a function.*

I have learned several ways to think about this relationship called a function. One way is to describe this relationship of variables as a relationship between sets as shown in Figure 13. If for any member of one set (Set A) you can describe the predictability that enables you to tell with certainty the corresponding member of a second set (Set B), the relationship between the two sets is a function. In other words, given f (x) = y. For any x (x_1, x_2, x_3, . . .), you can predict any y (y_A, y_B, y_B, . . .).

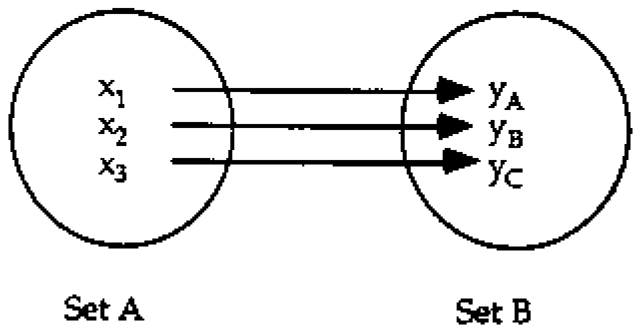


Figure 13-A FUNCTIONAL RELATIONSHIP BETWEEN TWO SETS

Leinhardt, Zaslavsky, and Stein (1990, p. 27) offer a similar explanation of this relationship using slightly different language. A function is a "special type of relationship or correspondence, a relation with a rule that assigns to each member of Set A exactly one member of Set B." A well-defined function will allow one to say what y value will go with a particular x. For each x there is only one value of y. Each element in the domain (Set A) corresponds to only one element in the range (Set B). When starting with a particular value, you can predict what the output will be with certainty. Functions carry with them the notion of predictability and regularity, the essence of pattern.

Why Spend Time on Functions? Understanding the composition of functions encourages students to build flexibility into the way they look at numbers. Studying functions in elementary school is a way to prepare for algebra or "the study of operations and relations among numbers through the use of variables" (Karush 1989, p. 4). It is also a way to prepare for operating "with concepts at an abstract level and then applying them, a process that often fosters generalizations and insights" (NCTM 1989, p. 150). CSMP is a curriculum oriented around functions. It attempts to build some conceptual understandings of algebra without using terminology through work with arrow roads. From this foundation, the expectation is that students in later grades will be able to do the kind of abstract generalizing required in the explicit study of algebra. A major goal is to build the kind of thinking and flexibility that will make a transition to the more advanced world of mathematics easier, especially aspects of mathematics having to do with functions.

The overall goal of CSMP and this lesson was to learn to look at calculation in different ways. Composing functions offers an alternative way of thinking about computation through the practice of mental strategies for the purpose of combining different numbers. For example, adding 12 can be thought of as adding 10 plus 2. A developer of CSMP admitted that the problems given in the CSMP teacher's guide, + 10 followed by + 2, + 7 followed by - 3, and + 9 followed by - 4 are not particularly strong examples of good practice in mental arithmetic. The numbers are easy to calculate and you do not have to have very complex mental strategies to figure them out. A more difficult example might be to add 38 and think of it, for example, as the composition of two functions, + 40 followed by - 2.

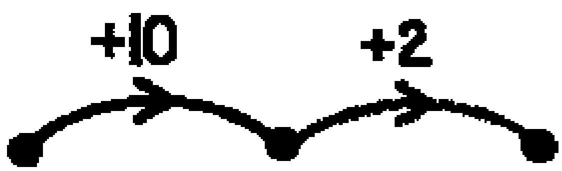
Looking back on these problems now, I can see how mental arithmetic could be emphasized, but this was not apparent to me as a goal when I read through the teacher's guide. When I asked students to explain how they figured out starting and ending numbers, it could have been understood as an attempt to make public students' mental strategies for figuring out these problems. The calculations they did, however, were not too difficult. Therefore, the mental strategies that students revealed were not too interesting for a student to report or the rest of us to consider. Perhaps larger and more difficult numbers to compose would have made completion of the table more interesting and the goal of building flexibility in mental strategies more obvious to me.

Learning the Purpose of the Task

What is the meaning of the task of searching for patterns in a table? Why are tables useful in learning about functions? What were students supposed to notice when they looked for patterns in the table? These were not questions I asked myself at the time I was teaching. At that point in time, I assumed that by doing the lesson, students would learn something. Rather, I hoped they would. Someone else had decided it was a worthwhile task. Three years later, I decided, once again, to find out for myself what was important about it. I approached my investigation of the meaning of this task with two questions. First, what sort of representation of a function is a table? Second, what does it mean to look for patterns within such a table? The problems under the heading of "composition of functions" in CSMP looked like the problems in Figure 14. The labels of the arrows varied. Once the table was filled with numbers, the students were to notice patterns. My current understandings of the purpose of the search for patterns within

starting number	ending number

Figure 14- THE FIRST PROBLEM I GAVE STUDENTS



starting number	ending number
99	111
8000	8012
250	262
1	13
7	19
203	215
~~4000~~ 4988	5000

Figure 15-THE BEGINNING AND ENDING NUMBERS GENERATED BY STUDENTS

this problem come from my own investigation into its meaning by using resources outside of the CSMP teacher's guide.

What were students learning about functions when they completed the table with the numbers as shown in Figure 15? To complete the table, students chose inputs and predicted outputs or chose outputs and predicted inputs. Picking any starting number, they could be certain of the ending number by adding 12 to it. They could also predict the starting number

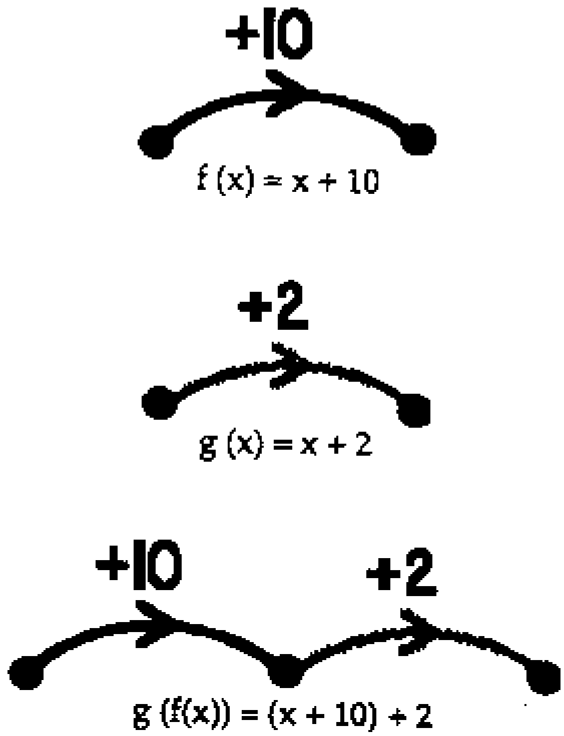


Figure 16-THE COMPOSITION OF FUNCTIONS REPRESENTED ALGEBRAICALLY

by subtracting 12 from any ending number. The only uncertainty came if a student had difficulty performing the addition or subtraction. When students started with the ending number, they performed the inverse function, arrived at by subtracting first 2 and then 10. Doing enough of these, students could have learned about inverse relationships among functions. Commutativity is also a possible focus. The students could have asked the question: If the two functions are applied to a starting number in the opposite order, do you get the same ending number? In other words, if I do + 2 followed by + 10, do I end up in the same place as + 10 followed by + 2? The table is normally thought of as the simplest representation of a function. In the example of + 10 followed by + 2, the table represents a composition of functions. The first operation, add 10, gives an output that becomes the input for the operation, add 2. The first function is f (x) = x + 10. The output of this function is symbolized by f (x). The input is x. The second function is g (x) = f (x) + 2 where f (x) is the input for g (x). It can be written algebraically as show in Figure 16. In the case of the composition of functions, the two functions are composed such that the outputs for the first function become the inputs for the second function.

Looking for the kinds of patterns that lead to an understanding of functional relationships involves multiple steps as well. The two columns of a table come in rows which have a beginning number and an ending number. The beginning number determines the ending number because the labels of the arrows allow me to predict an ending number from a beginning number or vice versa. To notice patterns as they relate to functions, I begin by making observations about a pair of numbers in one row. If these observations hold true for more than one row, they are patterns.

There is another layer of sophistication to the analysis of patterns in tables. Looking, for example, at a set of starting and ending numbers not chosen randomly—in the example, Figure 17, the starting numbers are all multiples of 3—I can look for patterns down the columns as well as across the rows. By putting the beginning numbers in order and choosing them with some regularity, I can make use of patterns in the column of starting numbers to predict the column of ending numbers. Since the outputs depend on the inputs, I can look to see if there are patterns in the ways that the outputs change when there is a certain orderly change in the inputs. I could ask, for example, Is there a constant difference between the outputs when there is a constant difference between the inputs? Patterns in a

starting number	ending number
3	15
6	18
9	21
12	24

Figure 17-STARTING AND ENDING NUMBERS AS MULTIPLES OF 3

table where the numbers are ordered and not randomly chosen have the potential to be more complex because they represent a coordination between changes in inputs and outputs.

Patterns and functions are also a way to consider classification. Functions are objects and, as objects, they can be classified and sorted. Different functions will have different kinds of patterns. For example, the function + 12 has certain patterns. The functions + 10 or + 2 will have other patterns.

T: Look closely at this chart. What patterns do you notice?
S: An ending number is always larger than the starting number.
S: If you start with an even number, you end with an even number. If you start with an odd number you end with an odd number.
S. An ending number is always 12 larger than the starting number. (P. 14)

Figure 18 STUDENT RESPONSES TO THE QUESTION OF PATTERNS FROM CSMP

Learning to Make Sense of the Search for Pattern

What sense can be made out of the question of looking for patterns in the context of a table of numbers? To answer this question, I look at both the responses that are a part of this lesson in the teacher's guide and students' responses made during the lesson. How did I understand these responses in the teacher's guide then, and how do I understand them now? What sense did I make out of the students' responses then, and how do I interpret them now? What do the inconsistencies between my perceptions and understandings then and now say about what I am learning?

Responses in the Teacher's Guide. If the CSMP dialogue is intended to be a guide and a way to prepare for what might happen along the way, an understanding of the text seems important. What I found is that just reading the text did not guarantee that I understood it the way CSMP may have intended. In my explorations of the intentions of CSMP, I discovered a wealth of mathematical ideas and connections within the teacher's guide that were not apparent to me at the time and only recently became visible to me in a conversation with a math educator involved in curriculum development and familiar with the development and design of CSMP.

I focused my conversation with a developer of CSMP on the three student responses listed in CSMP in response to the question of noticing patterns (Figure 18). When I taught the lesson, I was under the impression that these responses were ones I should expect from students because these represented how other students have responded to the question. What I have since discovered is that the student responses that appear throughout CSMP come either from the developers' experiences in classrooms with students, or they represent ideas that are mathematically interesting from the perspective of the person or people responsible for writing the text of the lesson. I learned that the responses in this particular lesson were mathematically interesting ones and not ones that I could necessarily expect from students. I wish that the teacher's guide had included information on where these responses had come from and why they were considered mathematically interesting.

Here is how I now understand the responses in CSMP. The third response in the text, that the ending number is always 12 larger than the starting number, is a response at the level of a generalization and the most related to the function notion. This response represents an important mathematical idea through its description of the relationship between the starting and ending numbers. The first response, that the ending number is always larger than the starting number, is an observation that is a more general variation on the third response. The second response, if I start with an even number, I end up with an even number, or the parity of the odd/even relationship of the starting number and ending number, is an interesting observation. If I add 11 rather than 12, the parity would be different. The significance of each response and how they are related to one another are things I have learned but not from the teacher's guide. If I had known any of this going into the lesson, would it have helped me along the way? At this point I can only hypothesize. I think the answer to this question would be yes.

The questions of how much, what kind, and in what ways to provide information for the teachers create ongoing dilemmas in curriculum development, and they are problems that the developers of CSMP have considered. When I voiced my concern of not having had access to certain information and ideas through the teacher's guide, the CSMP curriculum developer explained concerns about giving too much background on the responses which might make the use of the guide too tedious for teachers. What is the balance between enough and too much information in a teacher's guide? In this particular example, how useful are the suggested students' responses without the additional information

I learned on my own? If I had had some understanding of these responses, I think I would have had a better sense of what made for sensible answers. I expected the students' responses to vary, but without an understanding of the significance of the mathematics, I had no idea what reasonable variations might be.

What were students seeing? What am I able to see and hear in students' responses now that I was unable to see or hear at the time I was teaching? I was curious to see if it really was the case that what students had said in response to the question, "Do you see any patterns?" was unlike any of the students' predicted responses in the teacher's guide. In examining this lesson three years later, I have gone back to the video tape of this lesson and replayed it with the scripted lesson from CSMP in one hand and a transcript in the other.

Valerie's Response. Figure 19 is an excerpt from the teacher's guide placed side by side to the excerpt from the lesson's transcript where I asked the question of patterns.

As I look at the teacher's guide and the transcript, I see that Valerie, the first person to respond to my question of patterns, gave one of the answers that is in the teacher's guide. I even responded to her, added the composition arrow, and labeled it with plus 12. Yet, I continued with what resulted in a pointless discussion of patterns, and I ended that class feeling frustrated that students had not responded in any of the ways predicted by the teacher's guide. What does this discovery of the similarity of Valerie's comment to what appears in the teacher's guide say about what I understood to be my frustrations at the time and *what I have learned since then that might help me to understand my feelings*? I wanted to try to understand rather than discount the sense I

The CSMP Script	**Transcript 9/20/89**
T: Look closely at this chart. What patterns do you notice?	Ms. H.: Okay, I want you to look at these numbers. Do you see any patterns? Valerie?
S: An ending number is always larger than the starting number.	Valerie: <u>Each of them have a beginning number, and then they have an ending number that is 12 more.</u>
S: If you start with an even number, you end with an even number. If you start with an odd number, you end with an odd number.	Ms. H.: Okay. And how do you know that it is 12 more?
S: <u>An ending number is always 12 larger than the starting number.</u> (P. 14)	Valerie: Because you have to add 10. You find a number; you add 10 to it, and then 2.
	Ms. H.: If I were to put in another arrow here, what would I put? Plus what?
	Valerie: Plus 12.

FIGURE 19-CSMP STUDENT RESPONSES COMPARED TO VALERIE'S RESPONSES

made out of the situation at the time. How could it be that I thought all students' responses to the question of patterns were meaningless and bore no resemblance to what was in the teacher's guide when Valerie's response was really such a close match?

One explanation that comes immediately to my mind was my focus on having a discussion. If this was the case, then I was not really focused on the right answer, and it might make sense that I did not "hear" Valerie or want to "hear" her. If I had, she was the first person to respond, and her "right answer" might have ended the discussion. What more was there to talk about? It also seems quite possible that I might have "heard" Valerie but did not really understand what she was telling me. At the time, I was uncertain about the point of the lesson. If the question of noticing patterns relates to the reasons why one would look for then, it might follow that I was listening for the responses in the teacher's guide, but I did not really understand their significance to the question of patterns in this context. I did not really know what I was listening for. My ability to hear Valerie now may have something to do with what I have learned about patterns and functions since that year of teaching.

Valerie's response, that each of the beginning numbers was an ending number that is 12 more, seems on the surface to be rather mundane. But I appreciate now ways in which it is mathematically significant. Her response is a generalization that defines the composition of the two functions in this problem. If I had asked her to write down what she was saying in the shortest form possible, she would probably have written something like, "number + 12." This could be thought of as the equation "x + 12." I now see that looking for patterns in a table is like looking for the equation that defines the function. This is a move in the direction of algebra. Valerie's generalization meets the criteria of regularity and predictability, the two key characteristics of patterns. It is a statement that holds true for the pairs of numbers in all rows. You could choose any starting number and predict with certainty, using Valerie's pattern, the ending number.

Other Students' Responses. What about the responses of the other students? Were those patterns? If not, what were they? What the other students were observing were not mathematically relevant patterns. They were observing regularities in the numbers. Take the "pattern" a student noticed as shown in Figure 20 for example—there are three zeroes in 8000 as well as 5000. There is nothing predictable about this observation. It does not tell me anything about what else is happening with numbers in the table. If I made use of the criterion that to find patterns I must start out with observations about a pair of numbers in one row, the responses where students made observations about pairs of numbers not in the same row are eliminated immediately as patterns. The "pattern" identified in the table of Figure 20 is an example of this. Even when

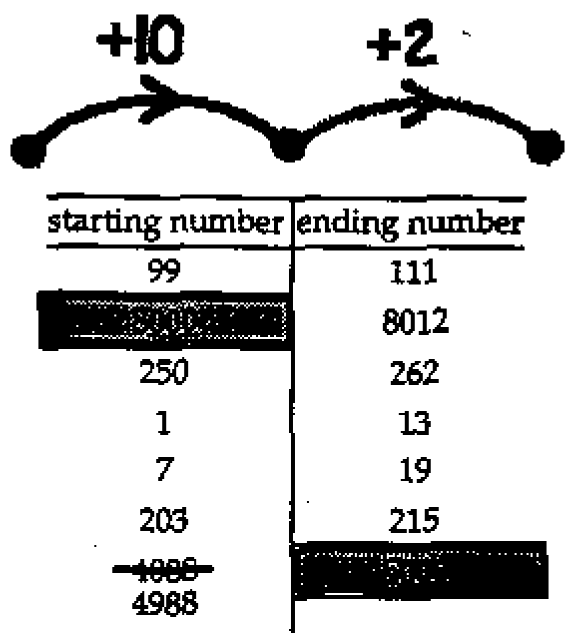


Figure 20-A STUDENT'S EXAMPLE OF A PATTERN

students were looking at pairs of numbers in the same rows, the sorts of regularities they were noticing were not predictive.

When students explained how they found beginning and ending numbers to fill in the table, they gave procedural explanations of how they got from one to the other. Their explanations, which I first interpreted as dull and pointless, reveal the meaning of the composition of functions and the relationship between outputs and inputs. The following examples illustrate this point.

Richard: Well, I did 99 plus 10 equals 109 and then plus 2 is like 9 plus equals 11.
Lucy: One plus 10 is 11 plus 2 is 13.
David: Because 7 plus 10 is 17 and 17 plus 2 is 19.
Bob: Because 203 plus 10 is 213, 213 plus 2 is 215.

If their procedures were represented in a chart of inputs and outputs, they would look as they appear in Figure 21.

Input	Output	Input	Output
99	109	109	111
1	11	11	13
7	17	17	19
203	213	213	215

Figure 21-THE RELATIONSHIP BETWEEN INPUTS AND OUTPUTS

Perhaps there was something interesting to notice in their responses to the question of how they got from their beginning number to their ending number. Mathematically, the composition of functions is a layer of complexity that is quite significant even though the mental arithmetic necessary to fill in the chart was rather simple for fourth graders.

Bob's question about what kinds of number they should choose as beginning numbers pointed out to me that there were no constraints on the numbers to use in the chart as the task was presented in the teacher's guide. When either the starting or ending number is chosen at random, as was the case in the problem I gave students, the chance of noticing interesting patterns is reduced. Constraining the choice of numbers and ordering the numbers chosen could help to make obvious some interesting patterns between rows and columns which might otherwise go unnoticed.

LEARNING THE SHAPE OF THE RIVER

Mark Twain's writing about what he had to learn to navigate the river in *Life on the Mississippi* ([1883] 1990) offers me images of what it is that I needed to learn as I attempted to make changes in my math teaching.[15] In this section, I construct parallels between his early learning experiences and mine mainly in the sense of his growing understanding of the nature of what he needed to learn and the usefulness of a text in doing the work.

In this particular lesson, I found myself dependent on the CSMP teacher's guide. I trusted that it was going to help me do the kind of mathematics teaching I envisioned. In a similar way, Twain began his adventures on the river with a notebook in his hands on which he depended. Twain, like me, thought this text was going to be the key to navigating the river. He wrote about what was in his notebook and his sense of its contents: "I had a notebook that fairly bristled with the names of towns, 'points,' bars, islands, bends, reaches, etc.; but the information was to be found only in the notebook—none of it was in my head" (p. 43). For both Twain and I, our work while trying to follow these guides did not go as planned. We both became frustrated. Within a short time on the river, Twain found that the information in his notebook was insufficient for the navigation he needed to do. I found the same to be true in my efforts to make use of the teacher's guide. Following its directions was not helping me teach in the way I imagined it would when I found myself trying to interact with students around mathematical ideas. Twain ([1883] 1990) wrote of the frustration his reliance on his notebook caused him:

[15]As was noted earlier, Twain's experiences offered me one set of images and language for helping me to understand and describe the process and particulars of learning to teach mathematics for understanding. In my dissertation, I play, additionally, with images and language of improvisational jazz and dance.

> The boat came to shore and was tied up for the night . . . I took my supper and went immediately to bed, discouraged by my day's observations and experiences. My late voyage's notebook was but a confusion of meaningless names. *It had tangled me all up in a knot every time I had looked at it in the daytime. I now hoped for respite in sleep; but no, it reveled all through my head till sunrise again, a frantic and tireless nightmare.*[16] (P. 48)

I can understand how he felt. The teacher's guide and the sense I made of it seemed to hinder my teaching more than help it. I was naive to think that the teacher's guide could carry aspects of this teaching that I was learning were my responsibility. For example, I needed to have an understanding of the mathematical purposes for what I was doing, a mathematical sense of why I was asking the questions I was asking. Simply following the script in the teacher's guide was not working. Without a sense of purpose or an understanding of what mathematics was important for my students to learn, I was lost as far as being able to figure out what to do next.

Twain learned much about navigating the river from Mr. Bixby, an experienced riverboat pilot on the Mississippi River and Twain's teacher. Mr. Bixby encouraged Twain to keep a notebook and intended for it to be useful. But how did he expect him to make use of it? What role did he think it should play in Twain's navigation of the river? What did the writers of CSMP know that made them think that what was in the teacher's guide would be helpful to me? What was it that either Twain or I needed to know to make use of our texts? Was it the text that needed to change or the way we made use of it? Or both?

I think that Mr. Bixby knew much more about the river than Twain realized, and I doubt that Mr. Bixby, an experienced riverboat pilot, ever imagined that Twain would rely so heavily on his notebook. In my case, I did not want to be so dependent on the textbook, but in the heat of the moment, in the face of uncertainty, I fell back on old and familiar ways of using a math textbook. I trusted it. I followed it without an understanding that was meaningful to me of what was important for students to learn and why. Like Mr. Bixby, I do not think the developers of CSMP ever expected me to cling so tightly to the text. That was part of my problem. But to stray from the text in purposeful ways, without wandering too far from important mathematics, I needed a much stronger and clearer sense of purpose for what I was doing. I started out searching for that when I planned the lesson. When I failed to turn

[16] Author Heaton's emphasis, not Mark Twain's.

up much that seemed useful, my past experiences pushed me to go ahead with the lesson anyway. That is when I found myself in trouble, unprepared and ill-equipped.

Twain ([1883] 1990) wrote about what he learned about the way he needed to know what was in his notebook:

> I have not only to get the names of all of the towns and islands and bends, and so on, by heart, but I must even get up a warm personal acquaintanceship with every old snag and one-limbed cotton-wood and obscure wood pile that ornaments the banks of this river for twelve hundred miles; and more than that, I must actually know where these things are in the dark. (P. 47)

The idea that he learned he needed a “personal acquaintanceship” with the river seems related to what I learned about the sense of purpose I needed to teach mathematics for understanding. The authors of CSMP had a much greater sense of what was important for students to learn and why particular concepts were important than was accessible to me in the way the teacher’s guide was constructed and I interpreted it. The information in Twain’s notebook was a representation of the river. My teacher’s guide was a representation of the mathematical terrain. Mr. Bixby knew the river. The authors of CSMP knew the terrain. What would Twain’s notebook look like if it represented what Mr. Bixby knew about the river in a way that was accessible to Twain? Could it be represented in a text? What would it look like for the teacher’s guide to represent what the developers of CSMP knew about the mathematics in a way that was accessible to me? What would it mean to design a textbook educative for teachers?

After Twain had been learning particulars on the river for a while, Mr. Bixby shared his view, in more general terms, about what it was that Twain was learning and why it was important: “You learn the shape of the river; and you learn it with such absolute certainty that you can always steer by the shape that’s in your head, and never mind the one that’s before your eyes” (Twain [1883] 1990, p. 54). From this particular lesson, I learned that I needed to develop a sense of purpose for what I was teaching that was my own—grounded in the mathematics others believe is important for students to learn but connected in meaningful and dynamic ways to my own mathematical understandings and what it is that particular students understand. I learned, specifically, about what it would mean to have this sense of purpose for teaching patterns and functions by looking closely at why patterns and functions are important to teach, why this particular lesson was important, and what students could learn from it. But what I have learned goes beyond the specifics of patterns and functions. What I take from this lesson is the need for a new sense of purpose, the questions to ask myself to go about constructing a sense of purpose for whatever mathematical idea I am trying to teach, and an understanding of why this sense of purpose is important in the kind of math teaching I am learning to do. *What I have learned is that the river or the mathematical terrain has a shape that I must learn to enable myself to navigate my way around in it. What is important is my understanding of the mathematics I am trying to teach for that will help me "hear" students and build connections from their ideas to the mathematics I want them to learn.*

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Mathematicians as well as mathematics educators have turned their attention to teacher's knowledge. In the last decade, researchers in mathematics education have begun to specify and assess teachers' knowledge in new ways [1]. At the same time, there is a general expectation that teachers have enough mathematical knowledge to teach. Pre-service teachersâ€™ challenges in presenting mathematical problems.Â The project addressed three major questions: What is the n ature of the mathematics courses prospective elementary teachers are required to take in their undergraduate education? What courses are required, who teaches them, who designs them, what is the content, how are they taught? An elementary mathematics teacher once argued that she and her students held four Rights of the Learner in the classroom: (1) the right to be confused; (2) the right to claim a mistake; (3) the right to speak, listen and be heard; and (4) the right to write, do, and represent only what makes sense.Â The purpose of this study was to better understand the knowledge and skills related to teaching of mathematics in-service teachers perceived as important for beginning teachers and what knowledge from their pre-service preparation they were able to apply in their instruction. Data was collected from practicing elementary and middle grades teachers who received their undergraduate preparation within the past eight years from a university in the Northeastern United States. The teacher poses questions that not only stimulate studentsâ€™ innate curiosity, but also encourages them to investigate further. Projecting a positive attitude about mathematics and about studentsâ€™ ability to â€œdoâ€ mathematics.Â In skills-based instruction, which is a more traditional approach to teaching mathematics, teachers focus exclusively on developing computational skills and quick recall of facts. In concepts-based instruction, teachers encourage students to solve a problem in a way that is meaningful to them and to explain how they solved the problem, resulting in an increased awareness that there is more than one way to solve most problems.