Modern Applications of Complex Variables: Modeling, Theory and Computation

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1 Overview of the Field

All professional mathematicians have taken an undergraduate class in complex analysis, covering analytic functions, singularities, residue calculus, conformal mapping, and so on. Indeed, one could argue that an early appreciation of the beauty of complex analysis, and of its power to solve a range of physically-arising problems, is a good indicator of a budding mathematician. Nonetheless, following a "golden age" after the Second World War, when complex variable theory (including integral transform methods) was used to solve a huge range of applied, mainly idealized, problems arising in fluid dynamics, elasticity and many other areas, complex analysis fell into disfavor among some applied mathematicians, who believed that all of the exciting discoveries had been made and that the subject had little new to offer. (Classic references from this period include the books of Noble, Sneddon, and Mittra & Lee [31, 37, 48].) In particular, the increasing prevalence of computers towards the end of the 20th century meant that many previously "in principle" calculations could now be done in practice: more complicated models could be quickly and cheaply solved numerically, obviating what some saw as the primary use for complex variable methods, the need for exact solutions to simplified models.

The range of speakers and topics gathered together for the 5-day workshop 15w5052 provided resounding evidence that the field of applied complex analysis is moving into a new era, burgeoning with new discoveries. The broad spectrum of subject areas they represented implies that complex analysis continues to play a central role in many areas of mathematics ranging from harmonic analysis to applied scientific computing. Particularly encouraging was the large number of early to mid-career mathematicians among the participants, whose work indicates that the field will be healthy for decades to come. The following sections of this report summarize the cutting edges of the subject that our Workshop considered, the specific outcomes of our Workshop, and the outlook for the future.

2 Recent Developments and Open Problems

The 21st century has brought a real renaissance in applied complex analysis, with a new generation of researchers using complex analysis in many different ways. Some of the developments that first led to the temporary decline in popularity of complex analysis as a key tool in applied mathematics are now responsible for its revival. For example, effective computational algorithms are being developed for finding conformal mappings [53] and for solving Riemann–Hilbert problems [40]. Researchers over the past decade or so have also cultivated an increasing appreciation of the role of complex singularities. There are physical examples (such as point vortices or dislocations) and there are mathematical examples: as Littlewood once remarked, the point of the Green's function is to make an infinity do positive work instead of being a disaster! In fact, the evolution and nature of the singularities of a system tell us much about the system itself. For example, the regularity and possible singularity formation in nonlinear PDE problems are conveniently studied by tackling the evolution of the singularities characterizing the extension of the solution to the complex plane [47]. Other good examples are Riemann–Hilbert problems, Painlevé transcendents, exponential asymptotics and Stokes lines (as arising in many different applications), conformal mapping applied to free boundary problems, and vortex dynamics – all of which made an appearance at our BIRS workshop. Again, modern computing power may be harnessed to great effect in studying these problems.

Progress has also been made recently on problems that were thought intractable. For example, within the past 10 years DeLillo *et al.* [15] and Crowdy [11] have made critical extensions to the classical Schwarz–Christoffel formula, extending its applicability from simply- to arbitrarily-connected domains. Such developments have paved the way for addressing a whole range of highly-relevant multiply-connected problems (for example in the manufacture of microstructured "holey" optical glass fibers). Moreover, new connections continue to be discovered between classical "complex variable" problems and other areas of mathematics: for example, the well-known Hele-Shaw free boundary problem is now known to be closely related to the theory of integrable systems, and to random matrix theory. The discovery of such connections stimulates new research and technology transfer between disciplines, and draws in a new set of researchers with different skill-sets.

3 Presentation Highlights

The theme of this workshop can be viewed as touching on three areas: new techniques in complex analysis, computational complex analysis, and modeling by means of complex analysis. The presentations covered all three areas. The workshop started with a biographical overview of the work and contributions of Alan Elcrat, by Tom DeLillo. Alan was a major figure in the field of applied and computational complex variable, and was one of the original organizers of the workshop. Most unfortunately, Alan passed away the week after our workshop proposal was accepted; his work was also honored in another workshop talk given by one of the organizers (Bartosz Protas).

Other presentations were divided into a number of groups, leading to an approximate path from more fundamental topics to applications. In addition, Elias Wegert gave a special evening lecture showing how complex functions can be usefully visualized using a variety of colored phase plots to illustrate different aspects of the function's behavior, in particular in the neighborhood of singularities. Apart from the valuable insight such plots afford, these plots are beautiful and mesmerizing, and examples feature in an annual calendar created by Prof. Wegert's group. His MATLAB-based *Complex Function Explorer* is freely available; further information may also be found in his book [51].

The other workshop presentations are summarized by theme below.

3.1 Conformal maps

Conformal mapping is very much a classical topic. Nevertheless, new results, new applications, and new computational methods continue to emerge.

Donald Marshall [29] spoke on recent work with Steffen Rohde on "conformal welding" and its application to planar graphs. His talk focused on computational aspects of this new application of conformal maps. Toby Driscoll described work with Everett Kropf on the creation of the Conformal Mapping Toolbox, an open-source, github-hosted project for the next generation of numerical conformal mapping software. Michael Booty described his joint work with Michael Siegel [5] showing how conformal mapping has been particularly useful in developing and validating a hybrid asymptotic-numerical method for solving problems of two-phase flow with soluble surfactant.

3.2 New developments in transform methods

There have been new developments with integral transforms. One that has attracted a lot of interest is the so-called "unified method" of Fokas, which is a method for solving linear boundary-value problems (and some nonlinear problems) [17, 14, 18]. Although realistic applications of this methodology are in their infancy, the organisers hoped that discussion at the workshop would be profitable: they were not disappointed.

Darren Crowdy described some of his recent work giving a new way of understanding the unified method, with applications to harmonic and biharmonic fields in complicated geometries, and involving boundary conditions of mixed type. Tony Davis presented his joint work with Darren Crowdy on the difficulties of using the Fokas method to study Stokes flow in an L-shaped channel; this is a canonical two-dimensional problem for the biharmonic equation. Bernard Deconinck (in joint work with Natalie Sheils) showed how the unified method could be combined with recent insights about interface problems [46] so as to derive fully explicit solutions of the time-dependent Schrödinger equation with piecewise constant potential.

3.3 Painlevé theory

The Painlevé equations are six nonlinear ordinary differential equations that have been the subject of much interest in the past forty years. They have arisen in a variety of physical applications, and they may be thought of as defining nonlinear special functions.

Peter Clarkson discussed special polynomials associated with rational solutions for the Painlevé equations and soliton equations [9]. He illustrated how these special polynomials arise in vortex dynamics [8]. Robert Buckingham presented joint work with Peter Miller in which rigorous asymptotic expressions for the large-degree behavior of rational Painlevé-II functions in the entire complex plane are derived [6]. Along the way, the Kametaka-Noda-Fukui-Hirano conjecture from 1986, concerning the pattern of zeros and poles, was confirmed.

Bengt Fornberg started his lecture by noting that the six Painlevé equations have a reputation of being numerically challenging. In particular, their extensive pole fields in the complex plane have often been perceived as "numerical mine fields". He then showed that, on the contrary, these pole fields provide excellent opportunities for fast and accurate numerical solutions across the complex plane. This was illustrated with several examples, from joint work with André Weidman [19, 20] and Jonah Reeger [44, 45]. Saleh Tanveer noted that, although there is much computation of Painlevé solutions, there are no methods to rigorously determine global error bounds. With Ali Adali, he used a recently developed method that was used to prove the Dubrovin conjecture [10] to determine approximate analytical expression for tritronquée solution for Painlevé I with rigorous bounds.

3.4 Riemann–Hilbert problems

As their name suggests, Riemann–Hilbert problems are classical. The basic problems are linear and scalar, but there are matrix and nonlinear versions.

Sheehan Olver reviewed several classical problems that can be reduced to Riemann–Hilbert problems, falling into three categories: integral representations, differential equations and inverse spectral problems. In all three cases, applying numerics to the Riemann–Hilbert problem allows for efficient approximation, that is uniformly accurate in the complex plane [40, 41, 42]. A particularly exciting aspect of these advances is the development of computational methods which achieve spectral accuracy, which is nontrivial given that the integral operators involved in the Riemann–Hilbert problem are singular [50]. Vladimir Mityushev solved a Riemann–Hilbert problem for circular multiply connected domains, and then used his results to explicitly write the effective conductivity tensor for regular doubly periodic arrays of cylinders [33, 34]. He then discussed some analogous problems for random arrays [32].

Alexander Minakov reformulated the Cauchy problem for the Camassa–Holm equation in terms of a vector Riemann–Hilbert problem so as to study the asymptotic behavior of the solution of the initial-value problem as $t \to \infty$, thus extending previous work [25]. Elias Wegert gave a survey of nonlinear Riemann–Hilbert problems with an emphasis on geometric aspects. In particular he addressed the existence and uniqueness of solutions for different classes of nonlinear boundary value problems and characterized solutions by extremal properties. He also discussed circle-packing problems [52] and the development of numerical methods.

3.5 Partial differential equations

Complex variables arise in various ways in the context of methods for solving partial differential equations.

Seung-Yeop Lee described his recent work with Roman Riser on the behavior of the two-dimensional Coulomb gas system, using large-degree asymptotic expansions of shifted Hermite polynomials [27]. Tom Trogdon presented joint work with Gino Biondini [4] on the Gibbs phenomenon for dispersive partial differential equations. They establish sufficient conditions for the classical smoothness of the solutions of linear dispersive equations for positive times, and they derive an oscillatory and computable short-time asymptotic expansion of the solution. Christopher Green presented his joint work with Jonathan Marshall [23] on constructing Green's function for the Laplace–Beltrami operator on a toroidal surface. It is written in terms of a single complex variable using two special functions: the Schottky–Klein prime function associated with an annulus, and the dilogarithm function.

John King described some model nonlinear parabolic problems, focussing on the implications of the nature of the complex singularities for real-line behavior such as blow up. This research direction is related to the question of global in time regularity of smooth solutions of important equations of mathematical physics, such as the 3D Navier-Stokes and Euler systems, a problem which still remains open [47]. Alexander Odesskii presented a simple construction of integrable Whitham type hierarchies [39]. André Weideman presented his work with Nick Hale on contour integral methods for the integration of elliptic partial differential equations on cylindrical domains. Such methods have been developed by I. P. Gavrilyuk and co-workers, but here the emphasis was placed on practical numerical considerations.

3.6 Applications: fluid mechanics

Conformal mapping, and related techniques of complex analysis, have long been applied to classical two-dimensional free boundary fluid dynamical problems such as the Hele-Shaw problem. But there are many other areas of fluid mechanics where the use and development of analytic function theory has been productive.

Chris Howls (in joint work with Jonathan Stone and Rod Self) explained how complex ray theory can be used to determine "cones of silence" in aeroacoustic applications [49]. Jon Chapman (with Chris Lustri, Phil Trinh and Jean-Marc VandenBroeck) considered free-surface potential flow in the limit of small Froude number. He explained how, in that limit, the surface waves are exponentially small, and arise via Stokes' phenomenon [28]. Jean-Marc Vanden-Broeck used numerical methods based on complex variables and series representations to solve a variety of two-dimensional potential free-surface flows, with special attention devoted to flows where the free surfaces intersect rigid surfaces.

Mike Siegel presented his work with David Ambrose in which they showed that a truncated system of equations for water waves that forms the basis of a widely-used numerical method is ill-posed. Their demonstration is based in an essential way on complex analysis. Chris Rycroft discussed his work with Martin Bazant on interfacial dynamics of dissolving objects in fluid flow. Their numerical method tracks the evolution of the object boundary in terms of a time-dependent Laurent series. Kostya Kornev (in joint work with Mars Alimov) introduced the problem of the capillary rise of a meniscus on substrates with complicated shapes [2]. He employed Chaplygin's hodograph transformation, and discovered that the contact line may form singularities even if the fiber has a smooth profile.

Bartosz Protas spoke about an unfinished project, studying vortex stability with Alan Elcrat. Their main focus was on a general approach to analyze the stability of inviscid flows with finite-area vortices [16], using methods of complex analysis. In order to handle the stability problem, new techniques of more widespread utility were devised to shape-differentiate singular contour integrals. Takashi Sakajo, in joint work with Rhodri Nelson and Tomoo Yokoyama, described the entrapment of force enhancing vortex equilibria in the vicinity of a Kasper wing [36].

Robb McDonald (work with A. Khalid, Jean-Marc Vanden-Broeck, Mark Mineev-Weinstein and Giovani Vasconcelos) considered unsteady propagating elliptical bubbles in an unbounded Hele-Shaw cell in the case of zero surface tension [24]. Numerical simulations demonstrate the important role played by singularities of the Schwarz function of the bubble boundary in determining the evolution of the bubble [30]. Scott McCue described his studies (with Bennett Gardiner, Michael Dallaston and Timothy Moroney) of the effect of a kinetic undercooling condition on the interface of an evolving bubble in a Hele-Shaw cell [21].

3.7 Other applications

Complex variable methods have a long history in other areas of mechanics, especially in the theory of elasticity. Indeed, it was these applications that motivated deep studies of singular integral equations over contours.

Anna Zemlyanova presented some of her work on systems of singular integro-differential equations in the context of a new model of fracture with a curvature-dependent surface tension [54]. The regularization and numerical solution of these systems was addressed and numerical examples were presented. Yuri Antipov discussed his recent work on singular integral equations on a segment with two fixed singularities [3]. The problem is reduced to a vector Riemann–Hilbert problem on the real axis with a piecewise constant matrix coefficient. As an application, the three-dimensional Dirichlet problem for the Helmholtz equation in the exterior of an infinite cone whose cross-section is a circular sector was solved.

Sonia Mogilevskaya used the Cauchy–Pompeiu formula to reduce integrals over an element arising in the three-dimensional boundary element method to integrals around the boundary of such an element, integrals that can often be evaluated analytically. This leads to fast and efficient algorithms [35, 43]. Andrew Norris gave a lecture on acoustic transparency and causality. It is well known that causality implies that the far-field pattern is analytic in the upper half of the complex ω-plane (ω is the frequency) together with other less well known properties. Implications of these general properties were explored, especially in the context of cloaking [38].

Yuri Godin presented his recent work on exact calculations of effective properties of periodic tubular structures. His approach is based on the construction of a quasiperiodic harmonic potential in the form of the Weierstrass zeta-function and its derivatives [22] Nick Trefethen presented joint work with Jon Chapman and Dave Hewett on a two-dimensional mathematical model of a Faraday cage. He showed numerical simulations, a theorem proved by conformal mapping, and a continuous model derived by multiple scales analysis.

4 Scientific Progress Made

With plenty of time available for informal interactions during the breaks and after the sessions, there were many opportunities for the workshop participants to engage in discussion aimed at addressing some of the open problems. While making measurable progress with such problems typically requires much longer time, below we highlight a few topics where some advances have been made.

One of the recurrent themes discussed during the workshop was the development of numerical techniques for the solution of the Riemann–Hilbert and related problems. Of particular interest are methods which achieve "spectral" accuracy (i.e., characterized by the approximation error vanishing exponentially with the refinement of the discretization). Such accuracy is however difficult to achieve given the singularity of the operators involved. Many of the discussions revolved around the presentation of Olver which featured results from his forthcoming monograph [50]. One question discussed at length was the extent to which such highly-accurate numerical methods can be generalized to other problems with a similar structure, such as various singular integral equations arising in the solution of elliptic boundary-value problems (these questions arose also in a number of other presentations).

A closely related topic which received a lot of attention during the workshop was development of software which can be used to solve such problems. An emerging direction in scientific computing, combining numerical and symbolic calculations in a hybrid approach, is represented by `Chebfun` [7], and two key members of the `Chebfun` development team, Toby Driscoll and Nick Trefethen, were present among the workshop participants. A number of discussions concerned how various singular integral operators can be efficiently implemented in `Chebfun`, which will greatly simplify the numerical solution of several problems at the heart of computational complex analysis.

On the more applied side, Llewellyn Smith and Protas were able to draw some interesting and promising connections between the recent work of Elcrat & Protas on the stability of Hill's vortex based on the shape-differential formulation and Llewellyn Smith's earlier studies of the same problem using asymptotic methods [26]. This connection may ultimately lead to a resolution of an open problem in theoretical fluid mechanics.

5 Outcome of the Meeting

It was evident from the atmosphere and discussions at the meeting that complex analysis in the 21st century is very much alive, vibrant with new theoretical developments, new computational algorithms, new applications, and new challenges, and crucially, with a new generation of mathematicians and physicists eager to address them (e.g. Tom Trogdon, Anna Zemlyanova, Chris Ryland, Chris Green, not to mention the graduate students of many participants who co-authored much of the work presented). The workshop brought together expert participants from different disciplines within Mathematics, Engineering and Physics, with their different perspectives contributing to the session discussions.

A specific session was held on the Wednesday evening (following the entertaining lecture by Elias Wegert on visualization of complex functions) to discuss aspects of the work presented thus far, as well as ideas for future collaborative opportunities. Several options were explored, including an application for a long program at IPAM (the Institute for Pure and Applied Mathematics at UCLA) and a thematic program at the Isaac Newton Institute in Cambridge, UK. Both ideas were received with enthusiasm, and the organizers are currently in discussions with the Director of IPAM, Russ Caflisch, to determine the feasibility of such a program. Two organizers (Protas and Cummings) also recently (March 2015) participated in an Applied and Computational Complex Analysis workshop held at Imperial College, London, organized by BIRS participants Darren Crowdy and Takashi Sakajo. From the discussions at that workshop it was evident that the BIRS workshop is already acknowledged as a landmark event in the recent development of complex analysis. At that workshop, the idea of an Isaac Newton program was discussed further. Crowdy may be willing to act as a local proposer and organizer of such a program.

It was apparent during the many discussion that all participants, and particularly the younger generation, found the BIRS workshop an extremely valuable experience. There are few opportunities to assemble such a critical mass of minds, with different but related perspectives, to use as a sounding-board. In addition, several collaborations were cemented or forged during the workshop, some of which were summarized in §4 above. Again, the younger participants in particular were able to meet for the first time with many researchers whose papers they had read, and took the opportunity to develop connections in informal discussions (the organizers themselves had several such discussions with younger participants; for example, Chris Ryland's talk drew on early work by Cummings & Kornev [13], and Cummings was able to direct Ryland to other work of relevance to his research [12]).

It was also apparent that many participants are engaged in solving physically-relevant problems which impact several other disciplines. Therefore we anticipate that the mathematical techniques that were discussed should ultimately become part of the arsenal deployed by scientists and engineers in solving many real problems in engineering, biological and medical sciences.

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Another application of complex variables is complex mathematical models: for example, when spectral analysis in economics is used, and the given economic data are regarded as random signals [7]. Yet more often economists face the situations when complex variables are used in mathematical modeling of discount cash flows [1; 4]. Models of computations find application in system design by their expression in modeling languages. The variety of system description languages mirrors the variety in mathematical models of computation. The finite-state machine (FSM) model is the cornerstone for constructing control systems.Â Applications are built on a model of computation, whether the designer is aware of this or not. Each possibility has strengths and weaknesses. Some guarantee determinacy, some can execute in bounded memory, and some are provably free from deadlock. Different styles of concurrency are often dictated by the application, and the choice of model of computation can subtly aifect the choice of algorithms. In theoretical computer science and mathematics, the theory of computation is the branch that deals with what problems can be solved on a model of computation, using an algorithm, how efficiently they can be solved or to what degree (e.g., approximate solutions versus precise ones). The field is divided into three major branches: automata theory and formal languages, computability theory, and computational complexity theory, which are linked by the question: "What are the fundamental capabilities and Complex variables and applications, eighth edition. Published by McGraw-Hill, a business unit of The McGraw-Hill Companies, Inc., 1221 Avenue of the Americas, New York, NY 10020. Copyright 2009 by The McGraw-Hill Companies, Inc. All rights reserved.Â That edition has served, just as the earlier ones did, as a textbook for a one-term intro-ductory course in the theory and application of functions of a complex variable. This new edition preserves the basic content and style of the earlier editions, the rst two of which were written by the late Ruel V. Churchill alone. The rst objective of the book is to develop those parts of the theory that are prominent in applications of the subject.